In the Cournot-Walras general equilibrium model, there may be "more to gain" by changing the numéraire, then by eliminating imperfections. A two-good economy example*

by

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Abstract

We give an example of an economy with two monopolists and consumer taxes, in which the removal of consumption taxes is shown to lead to a welfare decrease, while changing the way prices are normalized may lead to a welfare increase.


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1. Introduction

It is well-known among theoreticians, that in general equilibrium models with imperfect competition à la Cournot, price normalization matters: changing the numéraire has real effects. This is already discussed in Bronsard (1971, p. 102-103) and Gabszewicz and Vial (1972); the issue is further taken up in Dierker and Grodal (1986), who construct an example of a two-good economy in which an equilibrium (in pure strategies) exists if say, good 1 is chosen as numéraire, and fails to exist (even using mixed strategies) if the numéraire is good 2. See also Hart (1985) on existence issues and Böhm (1990) on normalization.

Imperfect competition is now entering the field of applied general equilibrium. The avenue was opened by Harris (1984), followed by many others. None of these papers discusses normalization, as if applied researchers considered that the "numéraire does not matter" principle extended without any problem from competitive to non-competitive equilibria.

In this note, we construct an example in which a simple change in the numéraire leads to "larger" welfare gains than going from an economy with consumption taxes to a tax-free economy. This may be a problem when one interprets the results obtained by applied non-competitive general equilibrium models: indeed, these usually exhibit much larger welfare gains from trade liberalization, than those obtained in similar experiments with competitive equilibrium models.

1 And even earlier than that, in a paper by Siamwalla and Schydlowsky as was pointed out to me by L. Gevers.
2 Berthélemy and Bourguignon (1989) set up a three-region (North, South and OPEC) model in which each region may be considered as having some monopoly power over the output of the commodity it produces; their algorithm fails to find an equilibrium when each region acts as a monopolist. According to F. Bourguignon, this is not due to the computer algorithm which fails to find solutions, but to non-existence of such a solution.
2. The model

We consider the following two-good economy.\footnote{The example consists of a modification of an example which was provided to me by L.-A. Gérard-Varet, who stimulated my interest in the subject.} There are two monopolists: each of them produces one single good and uses the other one as an input; returns to scale are constant; the production possibility set $Y_1$ of producer 1 (who produces commodity 1) consists of production plans $(y_{11}, y_{12}) = (1, -\alpha_1)y_1$ with $y_1 \in [0, 1]$; likewise, $Y_2$, the production set of producer 2 (who produces commodity 2) consists of plans $(y_{21}, y_{22}) = (-\alpha_2, 1)y_2$, with $y_2 \in [0, 1]$. Each monopolist ($i = 1, 2$) takes the other’s output and price as given and chooses his output level $y_i$ in such a way that:

\begin{equation}
(2.1) \quad y_i \in \arg\max (p_i - \alpha_ip_j)y_i, \quad i = 1, 2, \text{ and } i \neq j.
\end{equation}

There is one representative consumer whose utility function is:

\begin{equation}
(2.2) \quad u(x_1, x_2) = 4x_1 + 2x_2 - 2/3x_1^2 - 2/3x_2^2 - 1/2x_1x_2
\end{equation}

and whose endowment is $(0, 0)$; his income consists of the profits $\Pi_1$ and $\Pi_2$ of the two producers. Ad valorem taxes at rates $\tau_1$ and $\tau_2$ may be levied on his consumption bundle $(x_1, x_2)$; the total amount of taxes $T = \tau_1p_1x_1 + \tau_2p_2x_2$ is paid back to him under the form of a lump sum transfer; his budget constraint thus reads:

\begin{equation}
(2.3) \quad (1 + \tau_1)p_1x_1 + (1 + \tau_2)p_2x_2 = \Pi_1 + \Pi_2 + T.
\end{equation}

Finally, the market equilibrium conditions are:

\begin{align}
(2.4) & \quad x_1 + \alpha_2y_2 = y_1 \\
(2.5) & \quad x_2 + \alpha_1y_1 = y_2.
\end{align}
3. Definition of the Cournot-Walras equilibrium

The equilibrium prices \((p_1, p_2)\), together with the consumption bundle \((x_1, x_2)\) and the production levels \(y_1\) and \(y_2\), have to satisfy a certain number of conditions, which will be described now.

The consumer takes tax-ridden prices \((1 + \tau_i)p_i\) and income \(\Pi_1 + \Pi_2 + T\) as given and chooses a consumption bundle \((x_1, x_2)\) which maximizes his utility (2.2) subject to his budget constraint (2.3). To be an optimal consumption bundle, \((x_1, x_2)\) has thus to satisfy the following marginal condition:

\[
\frac{p_2(1+\tau_2)}{p_1(1+\tau_1)} = \frac{12 - 8x_2 - 3x_1}{24 - 8x_1 - 3x_2}.
\]  

Taking into account the two market equilibrium conditions (2.4) and (2.5), this leads to a system of three equations ((2.4), (2.5) and (3.1)) in four variables \((p_1, p_2, y_1 \text{ and } y_2)\) defining aggregate demands \(y_i(p_1, p_2), i = 1, 2\) or inverse demands \(p_i(y_1, y_2), i = 1, 2\).

Obviously, one of the four variables has to be fixed; this can be obtained, like in the pure Walrasian model, by normalizing prices. Since we deal with Cournot-behaving producers, we need inverse demands.

Normalizing first by setting \(p_1 = 1\), the inverse demand system is given by:

\[
\begin{align*}
\Pi_1(y_1, y_2) &= \left[1-\alpha_1 \frac{1+\tau_1}{1+\tau_2} \cdot \frac{12-(8-3\alpha_2)y_2-(3-8\alpha_1)y_1}{24-(8-3\alpha_1)y_1-(3-8\alpha_2)y_2}\right]y_1, \\
\Pi_2(y_1, y_2) &= \left[1+\tau_2 \cdot \frac{12-(8-3\alpha_2)y_2-(3-8\alpha_1)y_1}{24-(8-3\alpha_1)y_1-(3-8\alpha_2)y_2} - \alpha_2\right]y_2.
\end{align*}
\]
The Cournot-Nash behaviour of producers is, as usual, defined by a pair \((\bar{y}_1, \bar{y}_2)\) such that:

\[
\Pi_1 (\bar{y}_1, \bar{y}_2) \geq \Pi_1 (y_1, \bar{y}_2) \tag{3.6}
\]

\[
\Pi_2 (\bar{y}_1, \bar{y}_2) \geq \Pi_2 (\bar{y}_1, y_2), \tag{3.7}
\]

\[
y_i \in [0, 1]. \tag{3.8}
\]

A Cournot-Walras equilibrium can now be defined as a pair of prices \((\bar{p}_1, \bar{p}_2)\) and a pair of production levels \((\bar{y}_1, \bar{y}_2)\) satisfying (3.2), (3.3) and (3.6) to (3.8). Alternatively, such an equilibrium is a pair of prices \((\bar{p}_1, \bar{p}_2)\), a pair of production levels \((\bar{y}_1, \bar{y}_2)\) and a pair of consumptions \((\bar{x}_1, \bar{x}_2)\) satisfying (2.4), (2.5), (3.1) to (3.3) and (3.6) to (3.8).

Instead of setting \(p_1 = 1\), we can also normalize by setting \(p_2 = 1\), and accordingly, define an equilibrium, which may or may not turn out to give the same values for prices, productions and consumptions as the first one.

4. Equilibria of the model

Let us consider first the equilibrium in which \(\bar{p}_1 = 1\). We have to look for a solution of (3.4)-(3.6); the first order Kuhn-Tucker conditions have to be calculated, and the system of (in)equalities solved to find a Nash point. But it is easier to proceed as follows.

Firm 1’s decision is to compute \(y_1\) such that

\[
\bar{y}_1 \in \arg\max (1 - \alpha_1 p_2) y_1, \ y_1 \in [0, 1].
\]

Observe that, as long as \((1 - \alpha_1 \bar{p}_2) > 0\),\(^5\) the profit maximizing production schedule of monopolist 1 is \(y_1 = 1\), no matter what the second monopolist decides. Let us thus assume

\(^5\) This will become clear after \(p_2\) will have been computed.
this to be correct and set \( \bar{y}_1 = 1 \). The profit of the second monopolist is given by (3.5). Taking into account \( p_1 = 1 \) and \( y_1 = 1 \), it is easy to derive that the candidate Cournot-Nash solutions for the second monopolist are the roots of the following quadratic (in \( y_2 \)) equation:

\[
(1 + \tau_2)(16 + 3\alpha_1)[(1 + \tau_1)(9 + 8\alpha_1) - \alpha_2(1 + \tau_2)(16 + 3\alpha_1)]
+ 2(1 + \tau_2)(16 + 3\alpha_1)[(1 + \tau_1)(3\alpha_2 - 8) - \alpha_2(1 + \tau_2)(8\alpha_2 - 3)]y_2
+ (1 + \tau_2)(8\alpha_2 - 3)[(1 + \tau_1)(3\alpha_2 - 8) - \alpha_2(1 + \tau_2)(8\alpha_2 - 3)]y_2^2 = 0.
\]

(4.1)

Setting \( \alpha_1 = \alpha_2 = 0.1 \), the roots of (4.1) are given by:

\[
y_2 = \frac{251.02(1 + \tau_1) - 7.172(1 + \tau_2) \pm \sqrt{52187(1 + \tau_1)^2 - 1491(1 + \tau_1)(1 + \tau_2)}}{33.88(1 + \tau_1) - 0.968(1 + \tau_2)}.
\]

(4.2)

Assume now that \( \tau_1 = 0.5 \) and \( \tau_2 = 0.25 \); the profit maximizing solution for the second monopolist is \( \bar{y}_2 = 0.5846 \), with \( \bar{p}_2 = 0.2942 \); as can be checked, the pair \( (\bar{y}_1, \bar{y}_2) = (1, 0.5846) \) is a Nash equilibrium.

In the absence of taxes \( (\tau_1 = \tau_2 = 0) \), a similar calculation shows that \( \bar{y}_2 = 0.5679 \) and, consequently, \( \bar{p}_2 = 0.3606 \). Again, the pair \( (\bar{y}_1, \bar{y}_2) = (1, 0.5679) \) is a Nash equilibrium for the producers.

We now turn to the price normalization \( \bar{p}_2 = 1 \). Similar routine calculations lead to consider:

\[
\bar{y}_2 \in \arg\max(1 - \alpha_2 p_1) y_2,
\]

with \( \bar{y}_2 = 1 \) if \( (1 - \alpha_2 \bar{p}_1) > 0 \) and \( \bar{y}_1 \) is given by the roots of the following quadratic (in \( y_1 \)) equation:

\[
(1 + \tau_1)(4 + 3\alpha_2)[(1 + \tau_2)(21 + 8\alpha_2) - \alpha_1(1 + \tau_1)(4 + 3\alpha_2)]
\]
\[(4.3) \quad + 2(1+\tau_1)(4+3\alpha_2)[(1+\tau_2)(3\alpha_1-8) - \alpha_1(1+\tau_1)(8\alpha_1-3)]y_1 \\
+ (1+\tau_1)(8\alpha_1-3)[(1+\tau_2)(3\alpha_1-8) - \alpha_1(1+\tau_1)(8\alpha_1-3)]y_1^2 = 0.\]

As can be readily checked, (4.3) has only complex conjugate roots, so that the profit optimizing \( y_1 \) is on the boundary of \([0, 1]\), and will obviously be \( \bar{y}_1 = 1 \), whether consumption taxes are imposed or not. Setting as before \( \alpha_1 = \alpha_2 = 0.1 \), it is easy to calculate now that \( \bar{p}_1 = 5.5952 \) in the tax-ridden case \( (\tau_1 = 0.5, \tau_2 = 0.25) \), and \( \bar{p}_1 = 6.7143 \) in the absence of consumer taxes. In both cases, the pair \((\bar{y}_1, \bar{y}_2)\) is a Nash equilibrium.

Table 1 gives an overview of the prices and the allocations. The results clearly show that normalization matters: the two normalizations result not only in different relative prices, but also in different allocations (compare column 1 with column 3 or column 2 with column 4).

Moreover, when the first normalization is used \( (p_1 = 1) \), welfare decreases when taxes are removed; when \( p_2 = 1 \), welfare remains constant. The first result is counterintuitive for those used to think in terms of competitive markets, where removing the price distortion created by taxes should increase welfare. Here, we are in a non-competitive world in which we remove one imperfection (taxes), but keep the other one (non-competitive producers); and as is known from second-best analysis, everything can happen in that context.
Table 1
Results of the two normalizations

<table>
<thead>
<tr>
<th>Normalization</th>
<th>$p_1 = 1$</th>
<th>$p_2 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taxes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Prices</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>0.2942</td>
<td>0.3606</td>
</tr>
<tr>
<td>Consumptions</td>
<td>0.9415</td>
<td>0.9432</td>
</tr>
<tr>
<td></td>
<td>0.4846</td>
<td>0.4679</td>
</tr>
<tr>
<td>Productions</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>0.5846</td>
<td>0.5679</td>
</tr>
<tr>
<td>Welfare</td>
<td>3.7598</td>
<td>3.7488</td>
</tr>
</tbody>
</table>

It also appears that a "good way" to increase consumer welfare in our economy when there are consumer taxes is not to remove them, but to ... change the normalization rule.

5. Concluding comments

There are at least two important issues to which at least partial answers can be given. One is theoretical and concerns the consistency of the Cournot-Walras model; the second is empirical: does normalization matter in applied work?

Is the Cournot-Walras model proposed by Gabszewicz and Vial (and elaborated upon by many others since; see e.g. the survey by Hart (1985)) theoretically consistent? This model can be considered as a direct extension of the Walrasian one in which each producer (firm) maximizes profits at given prices; here, producers also maximize profits, but have some power on prices. In the Walrasian case, what matters is the (producing and
consuming) household whose activities can, since prices are given, be decentralized into pure consumer and pure producer activities (see e.g. Koopmans (1957)). Firms do not (and cannot) fool consumers who own them, and profit maximization by firms "comes along" with utility maximization by consumers.\(^6\) This can be easily seen by considering the Negishi (1960) format of a competitive equilibrium.\(^7\)

This convenient decentralizing property does not hold in the Cournot-Walras model, in which, on their "own behalf", firms (indirectly) set prices, take some surplus away from consumers who own them and prevent the economy from achieving a Pareto-optimal equilibrium. Why would consumers, be stupid enough to fool themselves?

This has recently led Codognato and Gabszewicz (1990) to reconsider the Cournot-Walras model as one in which some consumers have market power in the sense that they strategically set the share of the initial endowment they will offer and withhold the remainder for themselves. This setup can be extended without difficulty to economies with producers, as long as every noncompetitive firm is owned by a unique consumer\(^8\) who makes the selling decision, by maximizing his utility but not the firm's profit: at no point is the profit of a noncompetitive firm maximized. This is a theoretically satisfactory way of indirectly endowing firms with a utility maximizing behaviour.\(^9\) As a consequence, since profits of noncompetitive firms are not maximized, normalization will cease to matter.

One may think of avoiding the problem by normalizing prices after the first order conditions for profit maximization are taken and not before like is done in (3.4)-(3.5); this is unfortunately impossible here since, without further assumptions, prices cannot be

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\(^6\) This has been pointed out to me by Michiel Keyzer. See also Keyzer (1990).

\(^7\) This is best seen by considering Negishi framework to define an equilibrium. Negishi shows that (under some assumptions which are almost identical to those needed for the Arrow-Debreu proof) there exist welfare weights \(\alpha_i > 0\) such that the solution of the following mathematical program max \(\sum_i \alpha_i u_i(x_i)\) subject to \(\sum_j x_i \leq \sum_i o_i + \sum_j y_j, \ y_j \in Y_j, \ x_i \in X_i\). Profit maximization is implicitly obtained at every optimum (and thus at equilibrium), but does not appear explicitly in the formulation. In this program, \(u_i(x_i), \ x_i\) and \(o_i\) represent the utility function, the consumption plan and the endowment of consumer \(i\), \(X_i\) his consumption set, while \(y_j\) is the production plan of producer \(j\) and \(Y_j\) his production set.

\(^8\) Shared ownership can be taken care of by "aggregating" in some way those consumers who own the noncompetitive firm.

\(^9\) Endowing firms with a utility maximizing instead of a profit maximizing behaviour is also suggested by Jean Waelbroeck (1990).
solved out, profits cannot be expressed in terms of quantities only, and one cannot compute a Cournot equilibrium.\textsuperscript{10}

The second question which should be asked is whether normalization is a serious problem in applied Cournot-Walras models.

In the above example, producers who produce the two sole commodities traded in our economy are "big", and the power they have on prices is therefore large. In normalizing on one of the prices, one indirectly constrains the choice set of the producer, and this can be seen as "more serious" than just normalizing. In applied models, no non-competitive producer (or sector) will be as important and it is likely that the effects of normalizing on his or other producers' prices will not be too large.

Also, in most applications, some sectors at least will be competitive, and the normalization can be made on the price of a bundle of the latter goods; this does not \textit{per se} solve the normalization problem, but at least, it will not affect the choice set of the non-competitive producers. Moreover, as shown by Cripps and Myles (1988), this is the \textit{only} normalization which does not affect the behaviour of oligopolists.

For these two reasons, it may thus be that, in practice, the normalization issue is of no importance. This does not mean however that there is no problem: profit maximization (whether explicit, or implicit in the Marshall-Lerner conditions) by oligopolistic producers cannot be part of the specification of the model; firms can optimize only through decisions made by consumers.

\textsuperscript{10} See Keyzer (1990) on ways to achieve this, but in a framework which needs more assumptions on what is common knowledge to producers than the assumptions made in the Cournot-Walras model.
References


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