Acquiring foreign languages: 
A two-sided market approach*

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Abstract

We examine patterns of acquiring non-native languages in a model with two languages corresponding to two populations with heterogeneous learning skills, where every individual faces the choice of self learning the foreign language or refraining from doing so. We then introduce a profit-maximizing international school with two divisions (one in each population) so that each individual may now also choose to learn the other language at school. Comparative statics are analyzed in each model, and the solutions of both models are compared.

Keywords: Communicative benefits, linguistic equilibrium, learning costs

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1 Introduction

In this paper, we compare the outcomes of two different modes of learning languages, self-learning and learning at school, in a simple model of language acquisition, based on previous work by Selten and Pool (1991), Church and King (1993), Shy (2001), and Gabszewicz, Ginsburgh and Weber (2005). Selten and Pool (1991) formulate a general model of language acquisition. They introduce the notion of communicative benefits, that cover a wide range of economic, cultural and social advantages gained by learning languages. Church and King (1993) construct a game-theoretic model where every agent is proficient in a single language, but can acquire the other one at a cost which is identical for all agents. Every agent is faced with a binary choice: he can either learn the other language at a given cost, or refrain from acquiring it. The communicative benefit of an individual increases with the number of those with whom he/she can communicate using a common language. Thus, the equilibrium outcome depends on a network externality since the strategic decision by an individual to learn the other language expands the communication links for others who speak that language. The larger the number of individuals in the other language group who learn the native tongue of an agent, the smaller the benefit from second language acquisition for that agent. Church and King (1993) show that only corner solutions exist in equilibrium: either no one learns any language in either community (if the cost of learning is sufficiently high), or everybody learns the foreign language in one community while nobody does in the other. The fact that only corner equilibria exist is due to the assumption that learning costs are homogeneous over the population: once learning is beneficial for one agent initially endowed with some language, it is also so for all those who speak the same language. Ginsburgh, Ortuño-Ortín and Weber (2007) suggest that interior equilibria may exist, but focus on the empirical implications of the model, namely, the derivation of demand functions for languages. Gabszewicz, Ginsburgh and Weber (2005) construct a model where agents are heterogeneous in their language learning capacities, and this generates interior solutions.

Here, we build on the model by Gabszewicz, Ginsburgh and Weber (2005), who consider a simple case with two languages spoken in two countries or regions, and heterogeneous populations. Heterogeneity is introduced through the degree of language aptitude which leads agents to bear different individual learning costs. Agents opt to maximize their net communicative benefit defined as the difference between the communicative benefit discussed above,
and their individual cost of acquiring a new language. Gabszewicz, Ginsburgh and Weber require that in equilibrium expectations of learners in each population about the number of learners in the other are fulfilled.

In this paper, we first analyze the linguistic equilibria under self-learning only, and then extend the model by introducing a school of languages with one division in each population. The school is assumed to set its tuition fees to maximize its overall profit. This introduces a multi-sided market interpretation of the model in which tuition fees are explicitly taken into account when defining learning costs of individuals. The multi-lingual school then appears as a two-sided platform embarking individuals from both communities. While two-sided markets often involve positive cross network externalities, this is no longer the case in the bilingual context considered here. Agents are faced with negative externalities since the benefit obtained from learning the other language “at home” decreases with the number of agents who speak the home language in the other community.

Self-learning and learning at school are a priori very different mechanisms. The former introduces coordination among agents through the Nash equilibrium of a game with a number of players equal to the total number of citizens in the two communities. By contrast, the second mechanism requires the presence of an intermediary (the platform) to coordinate the decisions between the two sides of the market, and relies on profit maximisation and prices (tuition fees). In spite of these discrepancies, we show that both mechanisms lead to several similar features. For instance, under both mechanisms, an increase in the home community decreases the fraction of learners in the home community while the reverse holds when increasing the population in the other community. Also an increase in the learning cost, while always decreasing the fraction of learners in the smaller community, may have differentiated effects in the larger according to the value of this cost. Moreover, under both mechanisms, the fraction of learners is larger in the smaller population. Finally, the presence of a negative externality in language learning generates some results that may seem counterintuitive. For instance it may happen that the introduction of language schools in both communities reduces the number of learners in one of them with respect to the situation without schools.

1 An example of such a school is the well-known Berlitz School of Languages.
2 The notion was introduced by Rochet and Tirole (2003), and applied to several contexts such as credit cards (Wright, 2004) and the media industry (Gabszewicz, Laussel and Sonnac, 2001, Anderson and Coate, 2004).
The model is described in Section 2. Section 3 is to the two-sided market solution. Section 4 examines the comparatives statics of the model with self-learning only with the one when schools are introduced. Section 5 concludes and suggests further research avenues. Proofs of the propositions are relegated to an appendix.

2 The model

We consider two communities (regions, countries), $e$ and $f$, whose populations sizes are $E$ and $F$, respectively. All individuals are assumed to be born unilingual: initially, they only speak their native language, $e$ or $f$. However, they may consider acquiring the other language. This decision is based on two factors, potential benefits and learning costs.

**Benefits.** Following Selten and Pool (1991), we assume that the gross communicative benefit of an individual is represented by the number of others (in her own and in the other community), with whom she shares a common language. Assume that a proportion $l_f$ of citizens in $f$ learns language $e$. A citizen in population $e$ who refrains from learning language $f$ can communicate with $E$ fellow citizens and with the $l_f F$ individuals of the other population who have acquired language $e$. That is, her communicative benefit is represented by $E + l_f F$. If that individual from community $e$ learns language $f$, her communicative benefit will take into account all individuals from both populations $E + F$. However, to acquire the other language, she will have to face learning costs.

**Learning costs.** We assume that both communities consist of heterogeneous individuals uniformly ranked on the basis of a parameter $\theta \in [0, 1]$, which is the inverse of their ability to learn a foreign language. Those with lower $\theta$ can acquire a foreign language with a relative ease, and, in particular, an individual with $\theta = 0$ can learn it in her sleep. A larger $\theta$ is the sign of more difficulty in language learning. We identify individuals by their $\theta$.

We introduce two options of acquiring the foreign language. The first is self-learning. In this case we assume that every individual faces a personalized self-learning cost $C(\theta)$, which depends on her language ability. For simplicity, we assume that in both communities self-learning costs are given by $C(\theta) = c\theta$, where $c$ is a positive parameter common to both populations.\(^3\)

\(^3\)This assumption is introduced for the purposes of analytical simplicity only. The commonality of the cost parameter can be also challenged on empirical grounds, see Ginsburgh
The demand for foreign language learning creates an opening for language classes provided on a commercial basis. We assume now that individuals in both communities are offered an additional means of learning the other language by attending a profit-maximizing language school, which has a division in each community and charges tuition fees \( p_e \) and \( p_f \) to populations \( e \) and \( f \), respectively. While attending classes, individuals who learn still invest time and effort. We assume that an individual \( \theta \) in community \( e \) (\( f \)), who enrolls in the school will face the total cost of \( p_e + r\theta \), \( (p_f + r\theta \), respectively), where \( r \), again for reasons of analytical simplicity, is a common parameter in both communities. The school allows reducing the personalized cost so that the school-generated parameter \( r \) is smaller than the self-learning cost factor \( c \).

Assuming that the population size \( E \) is larger than \( F \), we impose the following assumptions on the parameters of the model:

\[
F < E < r < c,
\]

(1)

where the inequality \( E < r \) is a necessary and sufficient condition for a positive demand for schooling (see below). Note also that this inequality guarantees that the school-driven learning cost \( r \), and the self-learning cost \( c \) of the least able individual whose \( \theta = 1 \) in both populations, exceeds the maximum additional communicative benefit she can derive from learning the other language. This assumption ensures that not all individuals in \( e \) and \( f \) will learn the other language.

3 Demand for languages in a two-sided market

Individuals in both populations have the choice to learn by themselves, to attend the division of the school located in their community or refrain from learning the other language. Assume that the language school charges a tuition fee \( p_e \) in community \( e \), and that the proportion of individuals who learn language \( e \) in community \( f \) is given by \( l_f \). We derive the communicative benefit, or the total number of individuals, an agent \( \theta \) can communicate with, if she chooses either of three options.

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If she chooses the self-learning option, her communicative benefit net of costs is given by
\[ E + F - c\theta. \tag{2} \]
If she chooses to enroll in the language school the benefit is given by
\[ E + F - p_e - r\theta. \tag{3} \]
If she refrains from learning the other language she will be able to communicate with the members of her own community \(e\) and those in \(f\) who learn language \(e\), with no cost involved. Thus, her benefit (net or gross) will be
\[ E + l_f F. \tag{4} \]

For a given tuition fee \(p_e\) for learning language \(f\) in community \(e\), denote by \(\theta^1(p_e)\) the individual (if she exists) who is indifferent between self-learning and learning at school, i.e., for whom the benefits, given by (2) and (3), are equal:
\[ \theta^1(p_e) = \frac{p_e}{c - r}. \tag{5} \]
Note that individuals with \(\theta < \theta^1(p_e)\) prefer self-learning over learning at school, whereas all those with \(\theta > \theta^1(p_e)\) will make the opposite choice. Indeed, individuals with low \(\theta\) face low cost of self-learning and would not find it beneficial to pay tuition fees. However, to estimate the demand addressed to the language school, we have to make sure that individuals with higher \(\theta\) indeed enroll in the school and do not drop out of the language market altogether. For this, denote by \(\theta^2(p_e, l_f)\), the individual (if she exists) who is indifferent between learning at school and refraining from learning language \(f\). She is the individual for whom the benefits, given by (3) and (4), are equal:
\[ \theta^2(p_e, l_f) = \frac{F(1 - l_f) - p_e}{r}. \tag{6} \]
Individuals with \(\theta < \theta^2(p_e, l_f)\) prefer learning at school rather than forego learning language \(f\), whereas all those with \(\theta > \theta^2(p_e, l_f)\) prefer to refrain from learning. We restrict our attention to levels of tuition fees \(p_e\) and \(p_f\) that guarantee that the solutions are interior and that markets are not fully covered. We have the following result:
Lemma 1: Suppose (1) holds. Given the tuition fee $p_e$ and the fraction $l_f$ of learners of language $e$ in the community $f$, there is an incomplete market coverage and a strictly positive apprenticeship at the community $e$ only if

$$0 < \theta^1(p_e) < \theta^2(p_e, l_f) < 1.$$  \hfill (7)

In this case

(i) individuals with $\theta \in [0, \theta^1(p_e))$ engage in self-learning;
(ii) individuals with $\theta \in [\theta^1(p_e), \theta^2(p_e, l_f))$ enroll in the language school;
(iii) individuals with $\theta \in [\theta^2(p_e, l_f), 1]$ refrain from learning.

A similar reasoning applies to the community $f$ as well. That is, given the tuition fee $p_f$ and the fraction $l_e$ of learners of language $f$ in community $e$, there is an incomplete market coverage and a strictly positive apprenticeship at the community $f$ only if the following condition is satisfied:

$$0 < \theta^1(p_f) < \theta^2(p_f, l_e) < 1,$$  \hfill (8)

where $\theta^1(p_f) = \frac{p_f}{c - r}$, $\theta^2(p_f, l_e) = \frac{E(1 - l_e) - p_f}{r}$. Then

(iv) individuals with $\theta \in [0, \theta^1(p_f))$ engage in self-learning;
(v) individuals with $\theta \in [\theta^1(p_f), \theta^2(p_f, l_e))$ enroll in the language school;
(vi) individuals with $\theta \in [\theta^2(p_f, l_e), 1]$ refrain from learning.

Lemma 1 implies that the fraction of learners $l_e$ of the other language in the community $e$ is given by

$$l_e = \frac{F(1 - l_f) - p_e}{r}. \hfill (9a)$$

Similarly, we can determine the fraction of learners $l_f$ of the other language in community $f$

$$l_f = \frac{E(1 - l_e) - p_f}{r}. \hfill (9b)$$

\footnote{In fact, the fraction $l_e$ is determined by $l_e = \min\{(F(1 - l_f) - p_e)/r, 1\}$. However, since we restrict our examination to interior solution only, we proceed with the expression in (9a) and (9b).}
By solving the system of equations (9a)-(9b), we derive the fraction of learners in both communities as the functions of two tuition fees $p_e$ and $p_f$:

$$l_e(p_e, p_f) = \frac{r p_e - r F + E F - p_f F}{E F - r^2},$$  \hspace{1cm} (10a)

$$l_f(p_e, p_f) = \frac{r p_f - r E + E F - p_e E}{E F - r^2}.$$  \hspace{1cm} (10b)

Note that the Lemma allows us to derive the demand for apprenticeship in both populations, as only individuals from the intermediate range of $\theta$ will enroll in the language school. The demands in populations $e$ and $f$ are given by

$$E \left( \frac{F(1 - l_f) - p_e}{r} - \frac{p_e}{c - r} \right) \quad \text{and} \quad F \left( \frac{E(1 - l_e) - p_f}{r} - \frac{p_f}{c - r} \right),$$  \hspace{1cm} (11)

respectively. By substituting (10a)-(10b) in (11), we obtain the demands at the two divisions, $D_e(p_e, p_f)$ and $D_f(p_e, p_f)$. (The precise expressions for these and other derivations in this section are presented in the Appendix.) The school then chooses prices $\tilde{p}_e$ and $\tilde{p}_f$ which maximize total profits $\pi(p_e, p_f)$ over the two divisions:

$$\pi(p_e, p_f) = p_e D_e(p_e, p_f) + p_f D_f(p_e, p_f).$$

This function is concave (see Appendix), so that optimal prices $\tilde{p}_e$ and $\tilde{p}_f$ can be obtained from the first-order conditions. We shall show that the optimal prices yield positive demands for schooling, $D_e(\tilde{p}_e, \tilde{p}_f)$ and $D_f(\tilde{p}_e, \tilde{p}_f)$. Finally, we derive the equilibrium shares of learners in both populations, $\tilde{l}_e$ and $\tilde{l}_f$, by substituting optimal prices in (10a)-(10b).

## 4 Results

We now turn to our results which show how the number and fraction of learners in both populations are affected by shifts in parameters of the model. Through all our examination we shall assume that the inequality (1) holds.

**Proposition 2:** (a) An increase in the home population reduces the fraction of learners at home: the values of $\frac{d\tilde{l}_e}{dE}$ and $\frac{d\tilde{l}_f}{dF}$ are both negative.

(b) An increase in the population of the other population raises the
fraction of learners at home: the values of \( \frac{d\tilde{l}_e}{dF} \) and \( \frac{d\tilde{l}_f}{dE} \) are both positive
(c) An increase in the learning cost \( c \) reduces the fraction of learners in the smaller community \( F \). However, there is a threshold value \( \hat{c} \), such that the fraction of learners in the larger community \( E \) increases for \( c < \hat{c} \) and declines for \( c > \hat{c} \).
(d) The fraction of learners is larger in \( F \) than in \( E \).

**Proof.** See Appendix.

The intuition for these results, which are similar to those obtained when there is only self-learning are as follows: (a) shows that when there are more people to communicate with at home, the need to learn the language of the other community decreases; (b) an increase of the population on the other community makes it more compelling to learn their language; (c) when costs of learning increase, there are less learners; this is however not so in the larger community when the cost is not too high; (d) the small community gains more in learning the other language than the large one, since its population has access to more speakers.

It is useful to compare shares of learners in both communities examined in Proposition 2, with the situation where the schooling option does not exist and those who want to acquire the other language, have to rely on self-learning only.\(^5\) In the latter case we will search for individuals in both communities who are indifferent between self-learning and foregoing the study of the other language. The marginal type \( \theta \) of individuals who are indifferent between foregoing learning and self-learning is given by the cutoff values \( \theta(l_e) \) and \( \theta(l_f) \) in communities \( e \) and \( f \), respectively. Assuming that (1) holds, we equate the expressions (2) and (4) and obtain the expressions for shares of learners in both communities:

\[
\theta(l_f) = (1 - l_f) \frac{F}{c} \quad \text{and} \quad \theta(l_e) = (1 - l_e) \frac{E}{c}.
\]  

(12)

Since \( l_e = \theta(l_f) \) and \( l_f = \theta(l_e) \), by solving (12) we derive the equilibrium shares of self-learners:

\[
l_{e\text{sel}} = \frac{F(E - c)}{EF - c^2}; \quad l_{f\text{sel}} = \frac{E(F - c)}{EF - c^2}.
\]  

(13)

While comparing the equilibrium shares in (13) with \( \tilde{l}_e \) and \( \tilde{l}_f \), examined in the previous section, we reach the following interesting conclusions:

Proposition 3: (a) The fraction of learners with the school option exceeds the fraction of learners under self-learning in the smaller community $F$.

(b) There exists a value of self-learning factor $c^*$ (that depends on $r, E$, and $F$), such that the fraction of learners in $E$ with the school option is larger than the fraction of learners under self-learning if $c > c^*$. However, if $c < c^*$, the relationship is reversed.

Proof. See Appendix.

Proposition 3 leaves open the possibility that introducing schools increases the fraction of learners (with respect to self-learning only) in the smaller community $F$ while it reduces this fraction in $E$.

As illustrated in Figure 1 below for the case $c = 100$, $r = 80$, the introduction of schools increases the fraction of learners in both countries whenever the countries’ sizes are not too dissimilar and always increases the fraction of learners in $F$. It decreases the fraction of learners in $E$. In this case the direct positive effect of the introduction of schools is dominated by the indirect negative effect which introduced through the increase of the fraction of learners in $F$. 
Figure 1: The three areas in the $(E, F)$ space
In Figure 1 there are three areas. Starting from above and the left the first area where the fraction of learners decreases in the large community $E$ and increases in the small community $F$, the second, intermediate area where this fraction increases in both countries which are of similar sizes and, finally, the third area where the fraction of learners decreases in $E$ and increases in $F$.

5 Conclusion

Language learning is one among the rare examples of negative cross network externalities: the more people in a foreign community learn my own language, the smaller my own interest to learn theirs. Nevertheless, it constitutes a beautiful example of a multi-sided market: there exist language learning schools, with a division in several communities, in which it is possible to learn the corresponding foreign languages. Students pay tuition fees to facilitate their learning and these fees are collected in each division by the platform. Our paper compares the outcomes of two different language learning procedures. Learners proceed by using their own specific talents to learn the foreign language at a cost which varies within the community according to these talents. Alternatively, potential learners are helped by language professors, which reduces the individual learning cost, compared with learning in solo. However the subscription fee adds up to this reduced individual learning cost, which makes a priori unclear whether it is better to learn by oneself or by using the school’s services. According to the size of the tuition fees, some potential learners,– those with low individual learning costs,– prefer to learn by themselves, while those with higher learning costs may prefer to go to school.

The analysis leads to the following conclusions. First, the fraction of learners is larger in the smaller community. Moreover, an increase in the “home” population decreases the fraction of learners in the home community, whereas an increase in the population of the other community increases the fraction of learners at home. Similarly, an increase in the learning cost $c$ decreases the fraction of learners in the smaller community, while, in the larger one, this fraction first raises with the cost $c$ and then declines. Concerning the direct comparison of the total fraction of learners in each country with, and without schooling, we show that the introduction of the language school raises the number of learners of other language in the smaller commu-
nity. However, any conclusion about the larger community can be derived only by a careful examination of the parameters of the model (costs of learning under the two regimes and the population sizes of the communities).

While some of our conclusions crucially depend on the set of rather restrictive conditions, such as the linearity of the benefit function, our approach gives some insights into the mechanism of learning languages, in spite of the difficulties inherent to this problem.

6 References


Gabszewicz, J., V. Ginsburgh and S. Weber (2005), Bilingualism and communicative benefits, manuscript.


Before proceeding with the proofs, we need a few preliminary results.

The substitution of (10a)-(10b) in (11), yields, after some simplification, the system of demands at the two divisions:

$$ D_e(p_e, p_f) = E \left( \frac{rF - EF - rp_e + pfF - p_e}{r^2 - EF} - \frac{p_e}{c - r} \right). \quad (A1) $$

$$ D_f(p_e, p_f) = F \left( \frac{rE - EF - rp_f + peE}{r^2 - EF} - \frac{p_f}{c - r} \right). \quad (A2) $$

The total profit of the firm is $\pi(p_e, p_f) = p_e D_e(p_e, p_f) + p_f D_f(p_e, p_f)$. We need the following:

**Lemma 4**: The function $\pi$ is concave.

**Proof**: The Hessian matrix of the profit function can be presented as:

$$
\begin{pmatrix}
-\frac{E}{c - r} \cdot \frac{2cr - 2EF}{r^2 - EF} & \frac{2EF}{r^2 - EF} \\
2EF \cdot \frac{F}{r^2 - EF} & -\frac{F}{c - r} \cdot \frac{2cr - 2EF}{r^2 - EF}
\end{pmatrix}
$$

and the profit function is concave if:

$$ D_1 = -\frac{E}{c - r} \cdot \frac{2cr - 2EF}{r^2 - EF} \leq 0, $$

$$ D_2 = \frac{EF}{(c - r)^2} \left( \frac{2cr - 2EF}{r^2 - EF} \right)^2 - \left( \frac{2EF}{r^2 - EF} \right)^2 \geq 0, $$

where $D_1$ and $D_2$ are the relevant determinants extracted from the Hessian matrix. Since, under (1), $EF > cr$ and $EF > r^2$, $D_1$ is obviously
non-positive. After some simple algebraic transformations, the second determinant $D_2$ can be written as:

$$E^2F^2 + EF(c^2 + r^2) + c^2r^2,$$

an expression that is strictly positive. Thus, the profit function is concave. QED.

By Lemma 4, the optimal prices can be obtained from the first-order conditions:

$$\tilde{p}_e = \frac{F(c-r)(c-E)}{2(c^2-EF)},$$ (A3)

$$\tilde{p}_f = \frac{E(c-r)(c-F)}{2(c^2-EF)}. $$ (A4)

At these positive optimal prices, demands for schooling obtain as

$$D_e(\tilde{p}_e, \tilde{p}_f) = \frac{(r-E)EF}{2(r^2-EF)},$$ (A5)

$$D_f(\tilde{p}_e, \tilde{p}_f) = \frac{(r-F)EF}{2(r^2-EF)}. $$ (A6)

Under (1), these demands are positive. Equilibrium shares of learners are finally obtained by substituting optimal prices (A3) and (A4) in (10a)-(10b):

$$\tilde{l}_e = \frac{F(cr^2 - rEF - cEF + c^2F - c^2E - r^2E + 2E^2F)}{2(c^2-EF)(r^2-EF)},$$ (A7)

$$\tilde{l}_f = \frac{E(cr^2 - rEF - cEF + c^2F - c^2F - r^2F + 2F^2E)}{2(c^2-EF)(r^2-EF)}. $$ (A8)

Before we show that the solutions are, indeed, interior, let us observe, by taking the difference between (A8) and (A7) that:

$$\tilde{l}_f - \tilde{l}_e = \frac{(cr - EF)(c + r)(E - F)}{2(c^2 - EF)(r^2 - EF)},$$ (A9)

which is a positive number, by using (1).

**Lemma 5:** The values of the equilibrium shares of learners in both communities, $\tilde{l}_e$ and $\tilde{l}$, lie strictly between 0 and 1.
Proof: Consider community $e$. Since the denominator in (A7) is positive, the sign of $\tilde{l}_e$ is the sign of the numerator which is itself a second order polynomial $P(r) = A + Br + Cr^2$, where, by (1), $A = 2E^2F - c^2E - cEF < 0$, $B = c^2 - EF > 0$, and $C = c - E > 0$. It follows that $P(r)$ is increasing for $r > 0$. However, $P(E) = (c - E)(E - F)E > 0$ and since in our case, by (1), $r > E$, it follows that in the range of relevant $r$, the value of $P$ is positive, that shows that $\tilde{l}_e > 0$. By (A9), $\tilde{l}_f$ is positive as well.

To conclude the proof, observe that since equilibrium tuition fees are positive, equations (9a) and (9b) imply that both values, $\tilde{l}_e$ and $\tilde{l}_f$ are strictly smaller than one. QED.

Proof of Proposition 2:

Proof of (a)

\[
\frac{d\tilde{l}_e}{dE} = -\frac{1}{2} F \left( \frac{c(c - F)}{(EF - c^2)^2} + \frac{r(r - F)}{(EF - r^2)^2} \right),
\]
which is negative by (1).

Proof of (b).

\[
\frac{d\tilde{l}_e}{dF} = \frac{1}{2} \left( \frac{c^2(c - E)}{(EF - c^2)^2} + \frac{r^2(r - E)}{(EF - r^2)^2} \right),
\]
which is positive by (1).

Proof of (c).

The derivative of $\tilde{l}_f$ with respect to $c$, is

\[
\frac{d\tilde{l}_f}{dc} = \frac{E(2cF - EF - c^2)}{(c^2 - EF)^2}
\]
Since $F < E$ this derivative is always negative, as in this case the second-order polynomial $(2cE - EF - c^2)$ has no real root. However,

\[
\frac{d\tilde{l}_e}{dc} = \frac{F(2cE - EF - c^2)}{(c^2 - EF)^2}
\]
is positive for all values of $c \in [E - \sqrt{E^2 - EF}, E + \sqrt{E^2 - EF}]$ and negative elsewhere. Since, by (1), $c > E$, it follows that the above derivative is positive for $c \in (E, \hat{c})$, equal to 0 when $c = \hat{c}$ and negative for all $c > \hat{c}$, where $\hat{c} = E + \sqrt{E^2 - EF}$.

**Proof of (d).** This follows immediately from (A9). QED.

**Proof of Proposition 3.**

Proof of (a)

The difference $\tilde{l}_f - l_f^{self}$ obtains as

$$\tilde{l}_f - l_f^{self} = \frac{(cr - cE - rE + EF)(c - r)F}{2(r^2 - EF)(c^2 - EF)},$$

which is, by (1), strictly positive.

Proof of (b)

A similar reasoning for $\tilde{l}_e - l_e^{self}$ implies

$$\tilde{l}_e - l_e^{self} = \frac{(cr - cF - rF + EF)(c - r)E}{2(r^2 - EF)(c^2 - EF)}.$$

Since the denominator is again positive, the sign of $\tilde{l}_e - l_e^{self}$ is determined by the sign of the numerator, which is positive if and only if

$$c > c^*, \text{ where } c^* = E \frac{r - F}{r - E} > E.$$

Thus, the introduction of the language school increases the number of learners in $\tilde{e}$ when $c > c^*$. However, if $c < c^*$, the total number of learners $\tilde{e}$ will, due to externality effect, go down. QED.