Lecture 4. Choice under Uncertainty I

Now: choice has uncertain consequences

1. Expected Utility Theory

1.1. Lotteries

N: finite number of possible outcomes, denoted by $1, 2....N$

Definition: A simple lottery L is a probability distribution over N, i.e. $L = (p_1, p_2...p_N) \in \mathbb{R}^N_+$ with $\sum_N p_n = 1$

If L is chosen, the agent gets with probability p_n outcome n.

$(N-1)-$ dimensional simplex:

$$\triangle^{N-1} = \{p \in \mathbb{R}_+^N : \sum_n p_n = 1\}$$

$L \in \triangle^{N-1}$

Definition: Given K simple lotteries $L_k = (p_1^k, p_2^k ... p_N^k)$, $k = 1...K$ and an element α of the $(K-1)$-dimensional simplex, the compound lottery $(L_1..L_K, \alpha_1...\alpha_K)$ is the risky alternative that gives with probability α_k the simple lottery L_k.

Reduced lottery: Given a compound lottery $(L_1..L_K, \alpha_1...\alpha_K)$ the corresponding reduced lottery is a simple lottery L with $p_n = \sum_{k=1}^{K} \alpha_k p_n^k$.

Compound lotteries are represented by the corresponding reduced lotteries $\Longrightarrow$ analysis restricted to simple lotteries.

1.2. Preferences and utility functions over lotteries

Choice set: $\triangle^{N-1}$ - $(N-1)-$ dimensional simplex

Preferences $\succeq$ over $\triangle^{N-1}$ assumed to be complete and transitive (rational)

Properties of $\succeq$:

Continuity: $\succeq$ is continuous, if for all $L, L', L" \in \triangle^{N-1}$ the sets $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succeq L"\}$ and $\{\alpha \in [0,1] : L" \succeq \alpha L + (1-\alpha)L'\}$ are closed.

Independence: $\succeq$ satisfies the independence axiom, if for all $L, L', L" \in \triangle^{N-1}$ and $\alpha \in (0,1)$, we have:

$$L \succeq L' \text{ iff } \alpha L + (1-\alpha)L" \succeq \alpha L' + (1-\alpha)L"$$

Expected utility: The utility function $U : \triangle^{N-1} \rightarrow \mathbb{R}$ has expected utility form, if there is an assignment of numbers $(u_1, ...u_N)$ to the N outcomes such that for every simple lottery $L \in \triangle^{N-1}$ we have:

$$U(L) = u_1 p_1 + u_2 p_2 ... u_N p_n$$

$U(.)$ of expected utility form is also called Neumann-Morgenstern (v.N-M) utility function.

Proposition: Suppose that $U : \triangle^{N-1} \rightarrow \mathbb{R}$ is an expected utility function for $\succeq$ on $\triangle^{N-1}$. Then $\widetilde{U} : \triangle^{N-1} \rightarrow \mathbb{R}$ is another expected utility function for $\succeq$ iff there are scalars $\beta > 0$ and γ such that $\widetilde{U}(L) = \beta U(L) + \gamma$ for all $L \in \triangle^{N-1}$.

1.3. Expected Utility Theorem

Theorem: Suppose that $\succeq$ on $\triangle^{N-1}$ is rational, continuous, and satisfies the independence axiom. Then there are numbers $(u_1, ...u_N)$ assigned to the N outcomes such that for any two lotteries $L, L' \in \triangle^{N-1}$ we have:

$$L \succeq L' \text{ if and only if } \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

Proof: see MWG

Limitations of v.N-M utility function: Allais Paradox

2. Money Lotteries and Risk Aversion

Monetary lottery: Cumulative distribution function $F(x) : \mathbb{R} \rightarrow [0, 1]$

$F(x)$: probability that realized payoff is x or less.

$\triangle$: set of all distribution functions over interval $[a, +\infty)$

$u(x)$: assignment of utility values to amounts of money in the interval $[a, +\infty)$. Bernoulli function

v.N.-M utility function: $U(F) = \int u(x) dF(x)$

Definition: An agent is risk averse, if for any lottery $F(.)$ the agent weakly prefers the degenerate lottery, which gives $\int x dF(x)$ (i.e. the expected payment) with certainty, to the lottery $F(.)$. The agent is risk neutral, if he is for all lotteries indifferent between the lottery and the degenerate lottery which gives the expected pament with certainty. The agent is strictly risk averse, if he is only indifferent if $F(.)$ is degenerate itself.

Definition: For any $u(.)$ we define

i) the certainty equivalent of $F(.)$, denoted as $c(F, u)$, is given by the solution to:

$$u(c(F, u)) = \int u(x) dF(x)$$

ii) probability premium for any fixed amount x and any positive number ϵ, denoted as $\pi(x, \epsilon, u)$ is given by the solution to:

$$\begin{aligned} u(x) &= [\frac{1}{2} + \pi(x, \epsilon, u)] u(x + \epsilon) \\ &\quad + [\frac{1}{2} - \pi(x, \epsilon, u)] u(x - \epsilon) \end{aligned}$$

Proposition: Take an agent with a Bernoulli utility function $u(.)$ for money. Then the following properties are equivalent:

i) The agent is risk averse.

ii) $u(.)$ is concave.

iii) $c(F, u) \leq \int x dF(x)$ for all $F(.)$

iv) $\pi(x, \epsilon, u) \geq 0$ for all x, ϵ.

Measure of risk aversion: For a twice-differentiable Bernoulli utility function $u(.)$ the Arrow-Pratt coefficient for absolute risk aversion at x is given by:

$$r_a(x) = -u''(x)/u'(x)$$

$r_a(x)$ measures the "relative curvature "

Proposition: For two Bernoulli utility functions $u_1(.)$ and $u_2(.)$ the following statements are equivalent:

i) $r_{a2}(x) \geq r_{a1}(x)$ for all x

ii) $u_2(.)$ is a concave transformation of $u_1(.)$ ($u_2(.)$ is more "concave" than $u_1(.)$)

iii) $c(F, u_2) \leq c(F, u_1)$ for all $F(.)$

iv) $\pi(x, \epsilon, u_2) \geq \pi(x, \epsilon, u_1)$ for all x, ϵ

v) If $\int u_2(x)dF(x) \geq u_2(\overline{x})$ then $\int u_1(x)dF(x) \geq u_1(\overline{x})$ for any $F(.)$ and $\overline{x}$.

"$u_2(.)$ is consistently more risk-averse than $u_1(.)$".