Lecture 6: General Equilibrium - Definition and Welfare Properties

1. The Basic Model

L goods

initial endowments of the whole economy $\omega = (\omega_1 .... \omega_L)$

I consumers with consumption sets $X_i \subset \mathbb{R}^L$, rational preference relations $\succeq_i$

J firms with nonempty and closed production sets $Y_j \subset \mathbb{R}^L$.

Economy: $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega)$

Pure exchange economy: $Y_j \subset -\mathbb{R}_+^L$ for all $j = 1..J$

Firms and initial endowments belong to the consumers.

θ_{ij} is consumer $i's$ share of firm $j's$ profit, $\sum_{i=1}^{I} \theta_{ij} = 1$ for all j;

ω_{li} is $i's$ endowment of l, $\sum_{i=1}^{I} \omega_{li} = \omega_l$ for all l; $\omega_i = (\omega_{1i} ... \omega_{Li})$

Private Ownership economy

$$(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1} ... \theta_{iJ})\}_{i=1}^{I})$$

allocation $(x, y) = (x_1, ... x_{I,} y_1, ... y_J)$: consumption bundle for each consumer, production vector for each firm

Definition: An allocation is feasible, if for all $l = 1 \ldots L$, $i = 1 \ldots I$, $j = 1 \ldots J$:

$$\begin{aligned} x_i &\in X_i \\ y_j &\in Y_j \\ \sum_{i=1}^{I} x_{li} &= \omega_l + \sum_{j=1}^{J} y_{lj} \end{aligned}$$

Definition: A feasible allocation (x, y) is paretoptimal, if there is no other feasible allocation (x', y') such that:

$$x_i' \succeq_i x_i \text{ for all } i, \text{ and } x_i' \succ_i x_i \text{ for some } i.$$

Definition: An allocation (x^*, y^*) and a price vector $p^* = (p_1^*, ...p_L^*)$ is a *Walrasian equilibrium* of a private ownership economy $(\{(X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}...\theta_{iJ})\}_{i=1}^I)$, if the following holds:

i) Profit maximization: For each firm j, y_j^* fulfills:

$$p^* y_j^* \geq p^* y_j \text{ for all } y_j \in Y_j$$

ii) Optimal consumption: For each consumer i, x_i^* fulfills:

$$\begin{aligned} x_i^* &\succeq_i x_i \text{ for all} \\ x_i &\in \{x_i \in X_i : p^* x_i \leq p^* \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* y_j^*)\} \end{aligned}$$

iii) Market clearing: for all $l = 1...L$

$$\sum_{i=1}^{I} x_{li}^* = \omega_l + \sum_{j=1}^{J} y_{lj}^*$$

Definition: An allocation (x^*, y^*) and a price vector $p^* = (p_1^*, ...p_L^*)$ is a *price equilibrium with transfers* of an economy $(\{(X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \omega)$, if there is an assignment of wealth levels $(w_1 ... w_I)$ with $\sum_{i=1}^I w_i = p^*\omega + \sum_{j=1}^J p^* y_J^*$ such that:

i) For each firm j, y_j^* fulfills:

$$p^* y_j^* \geq p^* y_j \text{ for all } y_j \in Y_j$$

ii) For each consumer i, x_i^* fulfills:

$$\begin{aligned} x_i^* &\succeq_i x_i \text{ for all} \\ x_i &\in \{x_i \in X_i : p^* x_i \leq w_i\} \end{aligned}$$

iii) Market clearing: for all $l = 1...L$

$$\sum_{i=1}^I x_{li}^* = \omega_l + \sum_{j=1}^J y_{lj}^*$$

For the distribution $w_i = p^*\omega_i + \sum_{j=1}^{j} \theta_{ij}(p^* y_j^*)$, the *price equilibrium with transfers* of the economy $(\{(X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \omega)$ coincides with the *Walrasian equilibrium* of the private ownership economy $(\{(X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\omega_i, \theta_{i1}...\theta_{iJ})\}_{i=1}^I) \Rightarrow$

Walrasian equilibrium is a special case of the price equilibrium with transfers.

2. Welfare Theorems

First Welfare Theorem: If the preferences are rational and locally nonsatiated and if (x^*, y^*, p^*) is a price equilibrium with transfers, then the allocation (x^*, y^*) is paretooptimal.

Proof: Let (x^*, y^*, p^*) be a price equilibrium with transfers, and $(w_1 ... w_I)$ the associated wealth levels. By the optimality of the choice of the consumption bundle and local nonsatiation, it holds:

$$x_i \succ_i x_i^* \implies p^* x_i > w_i \text{ and } x_i \succeq_i x_i^* \implies p^* x_i \geq w_i$$

Assume that a feasible allocation (x, y) paretodominates (x^*, y^*). Then

$$\sum_{i=1}^{I} p^* x_i > \sum_{i=1}^{I} w_i = p^* \omega + \sum_{j=1}^{J} p^* y_j^*.$$

Because of profit maximization,

$$\sum_{j=1}^{J} p^* y_j^* \geq \sum_{j=1}^{J} p^* y_j.$$

Hence

$$\sum_{i=1}^{I} p^* x_i > p^* \omega + \sum_{j=1}^{J} p^* y_j$$

But feasibility of (x, y) implies $\sum_{i=1}^{I} p^* x_i = p^* \omega + \sum_{j=1}^{J} p^* y_j$

- a contradiction■

First welfare theorem implies, that the allocation connected to a Walrasian equilibrium is pareto-optimal.

Second Welfare Theorem: Take an economy $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega)$ with rational, locally non-satiated and convex preferences, convex consumption sets with $0 \in X_i$ for each i, and convex production sets. Then for every pareto-efficient allocation (x^*, y^*) there is a price vector $p^* = (p_1^*, ...p_L^*) \neq 0$ such that (x^*, y^*, p^*) is a price-equilibrium with transfers.

"Any pareoefficient allocation can be part of a Walrasian equilibrium for the appropriate redistribution of wealth".

3. Social Welfare Optima

Economy: $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega)$

Consumers preferences represented by continuous utility functions $u_i(.)$

Utility possibility set: $U = \{(u_1 ... u_I) \in \mathbb{R}^I$: there is a feasible allocation (x, y) with $u_i \leq u_i(x_i)$ for all $i\}$

Utility frontier UP: $UP = \{(u_1 ... u_I)\} \in U$: there is no $(u'_1 ... u'_I)$ such that $u'_i \geq u_i$ for all i and $u'_i > u_i$ for some $i\}$

Lemma: A feasible allocation (x, y) is paretoefficient iff $(u_1(x_1)...u_I(x_I)) \in UP$.

Social welfare function $W : \mathbb{R}^I \rightarrow \mathbb{R}$

Linear social welfare function: $W(u_1...u_I) = \sum_{i=1}^{I} \lambda_i u_i = \lambda u$

Maximization of linear social welfare (MLSW):

$$\underset{u \in U}{Max} \lambda u$$

Proposition: If $u^* = (u_1^* ... u_I^*)$ solves a MLSW with $\lambda >> 0$, then $u^* \in UP$. Furthermore, if U is convex, then for any $\widetilde{u} = (\widetilde{u}_1 ... \widetilde{u}_I) \in UP$ there is a $\lambda \geq 0$, $\lambda \neq 0$ such that $\widetilde{u}$ maximizes the linear scoial welfare function with weights λ.

If preferences and production sets are convex, any paretoefficient allocation can be characterized by maximizing a linear social welfare function