Lecture 9. Normal Form Games with Complete Information

Non-cooperative game theory: describes explicitly actions of agents (players)

1. Normal Form Game (Strategic Form Game)

Normal form game (strategic form game) defined by:

I : set of players (finite, until noted otherwise); typical element i

S_i : set of pure strategies of player i (finite, until noted otherwise); typical element s_i

$S = \prod_i S_i$: set of pure strategy combinations; typical element $s = (s_1, ...s_I)$

$s_{-i} \in S_{-i}$: combination of pure strategies of all players but i

$u_i : S \rightarrow \mathbb{R}$: payoff-function of i; $u_i(s)$ is $i's$ v.N.-M. utility if players choose strategy combination s

Typically assumed: common knowlegde of game and of rationality of all players

all players choose their strategies simultaneously

finite two person games: matrix representation

Mixed strategy σ_i : probability distribution over S_i

$\sum_i$: set of mixed strategies of i.

support of σ_i : set of all s_i with $\sigma_i(s_i) > 0$.

σ : profile of mixed strategies of all players

$\sum$: set of mixed strategies profiles.

σ_{-i}: profile of mixed strategies of all players but i

$$u_i(\sigma) = \sum_{s \in S} (\prod_{j=1}^{I} \sigma_j(s_j)) u_i(s)$$

Note: $u_i(.)$ is a continuous function over a compact set.

2. Nash equilibrium

Definition: A mixed strategy profile σ^* is a Nash equilibrium, if for all $i = 1, ...I$,and for all $s_i \in S_i$ it holds:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$$

"Every player plays a best reply to the other players strategy, and therefore no incentive to deviate."

Definition: A pure strategy profile s^* is a strict Nash equilibrium, if for all $i = 1, ...I$,and for all $s_i \in S_i \backslash \{s_i^*\}$ it holds:

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$$

"For every player a deviation from his strict equilibrium strategy would make him strictly worse off."

Nash-equilibrium requires common knowlegde of rationality and coordination

Examples: matching pennies, coordination game

Nash-equilibrium in pure strategies need not exist (matching pennies game)

Proposition: Every finite strategic form game has a mixed strategy Nash equilibrium.

Proof:

Step 1: Best response correspondence (reaction correspondence):

Best response correspondence of player i, $BR_i : \sum \rightarrow \sum_i$ with

$$BR_i(\sigma) = \{\overline{\sigma}_i \in \sum_i : \overline{\sigma}_i \text{ maximizes } u_i(\sigma_i, \sigma_{-i})\}$$

Note that for all $i, \sigma_i, \sigma'_i, \sigma_{-i}$ it holds:

$$BR_i(\sigma_i, \sigma_{-i}) = BR_i(\sigma'_i, \sigma_{-i})$$

Cartesian product of BR_i : $BR : \sum \rightarrow \sum$ with
$BR(\sigma) = (BR_1(\sigma), ...BR_I(\sigma))$

A fixed point of a correspondence $r : A \rightarrow A$ is $a \in A$ such that $a \in r(a)$.

A fixed point of BR is of course a Nash-equilibrium.

Step 2 (Kakutani's fixed point theorem):

A correspondence $r : A \longrightarrow A$ has a fixed point if:

i) A is a bounded and closed (compact), convex, nonempty subset of $\mathbb{R}^K$.

ii) $r(a)$ is nonempty for all a.

iii) $r(a)$ is convex for all a.

iv) $r(a)$ has a closed graph.

Step 3: BR has a fixed point.

i)$\sum$ is a closed, bounded, convex, nonempty subset of $\mathbb{R}^K$.

ii) $BR_i(\sigma)$ is nonempty for all σ, since it is the result of maximization of a continuous function over a compact set. Hence, $BR(\sigma)$ is nonempty for all σ.

iii) $BR_i(\sigma)$ is convex for all σ. For any $\sigma_i, \sigma_i', \sigma_{-i}$ with $u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma_i', \sigma_{-i})$ it holds for all $\alpha \in (0,1)$:

$$\begin{aligned} u_i(\alpha\sigma_i + (1-\alpha)\sigma_i', \sigma_{-i}) &= \\ \alpha u_i(\sigma_i, \sigma_{-i}) + (1-\alpha) u_i(\sigma_i', \sigma_{-i}) &= u_i(\sigma_i, \sigma_{-i}). \end{aligned}$$

Hence, if σ_i as well as σ_i' maximize $i's$ utility against σ_{-i}, any convex combination is also optimal.
Since the Cartesian product of convex sets is convex, $BR(\sigma)$ is convex for all σ.

iv) Assume BR does not have a closed graph. Hence, there is a sequence $(\sigma^n, \widehat{\sigma}^n) \rightarrow (\sigma, \widehat{\sigma})$ such that $\widehat{\sigma}^n \in BR(\sigma^n)$, but $\widehat{\sigma} \notin BR(\sigma)$, which implies $\widehat{\sigma}_i \notin BR_i(\sigma)$ for some i. Therefore, there exists $\epsilon > 0, \sigma'_i \in \sum_i$ such that $u_i(\sigma'_i, \sigma_{-i}) > u_i(\widehat{\sigma}_i, \sigma_{-i}) + 3\epsilon$.

Since $(\sigma^n, \widehat{\sigma}^n) \rightarrow (\sigma, \widehat{\sigma})$ and $u_i(.)$ is continuous, we have for large enough n

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\sigma'_i, \sigma_{-i}) - \epsilon > u_i(\widehat{\sigma}_i, \sigma_{-i}) + 2\epsilon > u_i(\widehat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon.$$

Hence, $u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\widehat{\sigma}^n_i, \sigma^n_{-i}) \Longrightarrow \widehat{\sigma}^n_i \notin BR_i(\widehat{\sigma}^n_i, \sigma^n_{-i})$, which is a contradiction.$\blacksquare$

Nash-equilibrium need not be unique: Example battle of the sexes

unique strict Nash equilibrium outcome need not be pareto-efficient: Prisoners' dilemma

Nash equilibrium in games with continuous strategy spaces

example: Cournot oligopoly

Proposition: Consider a normal-form game where every player i has a strategy space S_i that is a nonempty, compact, convex subset of an Euclidian space. If the payoff-function of every player is continuous in s and quasi-concave in s_i, then there exists a pure strategy equilibrium.

3. Iterated strict dominance

"Which strategies are save to use?"

Definition: A pure strategy s_i is strictly (weakly) dominated, if there exists $\sigma_i \in \sum_i, \sigma_i \neq s_i$ such that

$$u_i(\sigma_i, s_{-i}) > (\geq) u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Note: If s_i is dominated, then all mixed strategies with s_i in its support are also dominated.

Definition: Let $S_i^0 = S_i$ and $\sum_i^0 = \sum_i$. S_i^n denotes the set of pure strategies that survives the $n-th$ round of iterated deletion of strictly dominated strategies. It is defined recursively by

$$S_i^n = \{s_i \in S_i^{n-1} : \text{ there exists no } \sigma_i \in \sum_i^{n-1} :$$

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{n-1}\}$$

$$\text{and } \sum_i^n = \{\sigma_i \in \sum_i : \sigma_i(s_i) > 0 \Rightarrow s_i \in S_i^n\}$$

$S_i^\infty = \bigcap_{n=1}^{\infty} S_i^n$ is the set of pure strategies of i that survives iterated deletion of strictly dominated strategies.

Definition: A game is solvable by iterated strict dominance if S_i^∞ is a singleton for all i.

Examples: Dilemma Game, Cournot Oligopoly

Not all games are solvable by iterated strict dominance, e.g. matching pennies

4. Rationalizability

"What are all the strategies a rational player could play?"

Definition: Set $\widetilde{\sum}_i^0 = \sum_i$ and define recursively

$$\widetilde{\sum}_i^n = \{\sigma_i \in \widetilde{\sum}_i^{n-1} : \exists \sigma_{-i} \in \underset{j \neq i}{\times} \widetilde{\sum}_j^{n-1} \text{ such that}$$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \widetilde{\sum}_i^{n-1}\}.$$

The rationalizable strategies of i are $R_i = \bigcap_{n=0}^{\infty} \widetilde{\sum}_i^n$

Theorem: The set of rationalizable strategies is nonempty and contains at least one pure strategy for all players.

Theorem: Rationalizability and iterated strict dominance coincide for 2-player games.