

Lecture 11. Sequential Rationality in Games with Imperfect and Incomplete Information

1. Perfect Bayesian Equilibrium

Games with imperfect and incomplete information have information sets that consist of more than one node \Rightarrow

SPE is not enough to exclude noncredible threats.

Example: Noncredible threat game

2 NE: (L, L') and (R, R') .

Both NE 's also SPE 's, since the only subgame is the whole game.

However, NE (R, R') is implausible: To induce player 1 to choose R , player 2 must threaten to play R' , if 1 chooses L . However, if 2 actually has to make a choice, L' would always be better for 2 than R' - threat of R' is not credible \implies

Equilibrium should not only consist of strategy combination, but also of beliefs.

Requirements on the equilibrium

Requirement 1: At each information set, the player which moves must have a belief - a probability distribution - about which node in the information set he is. (For singleton information sets, the belief is of course trivial).

Requirement 2: Given their beliefs, the strategies must be sequentially rational - At each information set, the action taken by the player must maximize his payoff, given his belief and the subsequent strategy combination.

Requirement 3: The beliefs must be derived by Bayesian updating.

Noncredible threat game

For any belief of player 2 about where she is in her information set, she is best off by playing L' . Given this subsequent strategy "combination" and player 1's trivial belief about where he is in his own information set, player 1 is best off by playing $L \implies$

Only (L, L') fullfills Requirements 1 and 2.

For the noncredible threat game, it does not matter whether 2's beliefs are sensible or not - for any belief 2 should choose L' , requirement 3 not necessary. For other games, optimal actions do depend on the belief, so we have to ask what are sensible beliefs. \Rightarrow

Bayesian beliefs

Idea: beliefs that support an equilibrium strategy combination, must be consistent with this strategy combination

Noncredible threat game

equilibrium strategy combination $(L, L') \implies$ belief must be such, that player 2 thinks that she is in the left node of her information set, when her information set is reached.

To get "sensible beliefs" in general: Bayesian updating

p : profile of possibly mixed strategies

Examples from noncredible threat game

$$\text{E1: } p = \left(\frac{1}{2}L + \frac{1}{4}M + \frac{1}{4}R, R'\right)$$

$$\text{E2: } p = \left(\frac{1}{8}L + \frac{7}{8}R, L'\right)$$

$$\text{E3: } p = (R, R')$$

$\text{prob}(k | p)$: probability, that information set k is reached, when the players play p .

$$\text{E1: } \text{prob}(2' \text{'s information set} | p) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\text{E2: } \text{prob}(2' \text{'s information set} | p) = \frac{1}{8}$$

$$\text{E3: } \text{prob}(2' \text{'s information set} | p) = 0$$

$prob(x | p)$: probability, that node x is reached, when the players play p

$$E1: \quad prob(2's \text{ left node} | p) = \frac{1}{2} \dots prob(2's \text{ right node} | p) = \frac{1}{4}$$

$$E2: \quad prob(2's \text{ left node} | p) = \frac{1}{8} \dots prob(2's \text{ right node} | p) = 0$$

$$E3: \quad prob(2's \text{ left node} | p) = 0 \dots prob(2's \text{ right node} | p) = 0$$

$b_i(x | p, k)$: player i 's belief that he is in node x of his information set k , if the information set k is actually reached and the players play p .

Bayesian updating:

$$b_i(x | p, k) = \begin{cases} \frac{prob(x|p)}{prob(k|p)} & \text{if } prob(k | p) > 0 \\ \text{anything} \in [0, 1] & \text{else} \end{cases}$$

E1:

$$b(2's \text{ left node} \mid p, 2's \text{ information set}) = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

$$b(2's \text{ right node} \mid p, 2's \text{ information set}) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

E2:

$$b(2's \text{ left node} \mid p, 2's \text{ information set}) = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

$$b(2's \text{ right node} \mid p, 2's \text{ information set}) = 0$$

E3:

$$b(2's \text{ left node} \mid p, 2's \text{ information set}) = ?$$

$$b(2's \text{ right node} \mid p, 2's \text{ information set}) = ?$$

Definition: A Perfect Bayesian equilibrium consists of a strategy combination and beliefs that fulfill Requirements 1-3.

2. Signaling games

Two players, Sender and Receiver

Nature draws a type t_i for the Sender from a set of feasible types T according to probability distribution $q(t_i)$

Sender observes t_i , and then decides to send a signal m_j sends a signal from a set of feasible signals M .

Receiver observes m_j (but not t_i), and then chooses an action a_k from his action space A .

payoffs

$$u_S(t_i, m_j, a_k)$$
$$u_R(t_i, m_j, a_k).$$

This game used for the analysis of many different informational problems. E.g. Sender is a firm that needs money for a project. Investor uninformed about the profitability of the project, but firm can signal the profitability by its capital structure.

Other example: University Education as Job Market Signaling Device

Receiver: firm

Sender: employee

nature chooses whether a future employee is highly talented (type h) or not (type l); q is the probability of h , $0 < q < 1$.

The future employee chooses whether take a university education (m_e) or not (m_n).

The firm does not observe the talent of the employee, but his university education (or the lack of it).

There are in principle two positions available for the employee: One that gives a high salary and requires high talent, and a normal one for which the talent level is irrelevant. The firm decides about whether to put the employee on the high talent position (choice a) or on the normal position (choice b).

payoffs

Employee: The high talent job pays a salary of 4, the normal job a salary of 2. Successfully attending the university requires effort, which is higher for the l than for the h type. The monetary equivalent of these effort costs are 1 for the h and 3 for the l type.

Firm: For the high talent job, the firm's profit from employing an h type is 4. Employing an l type for this job is really a mistake, it gives no profits at all. University education increases the productivity of both types, such that profits increase by $\epsilon \geq 0$. For the normal job, neither a high talent nor the university make a difference. The profits are always 2.

Proposition: For all $\epsilon \geq 0$, $0 < q < 1$, a Bayesian perfect equilibrium $p^* = (p_S^*, p_R^*)$ is given by:

$$p_S^* = (m_e^h, m_n^l);$$

$$p_R^* = (a_e, b_n)$$

$$1 = b_R(h | \text{info set } e, p^*)$$

$$0 = b_R(h | \text{info set } n, p^*)$$

Proof:

1) Beliefs in accordance with Bayesian updating?

$$\text{prob}(\text{info set } e | p^*) = q$$

$$\text{prob}(\text{info set } n | p^*) = 1 - q$$

$$\text{prob}(h \text{ in info set } e | p^*) = q;$$

$$\text{prob}(l \text{ in info set } e | p^*) = 0;$$

$$\text{prob}(h \text{ in info set } n | p^*) = 0;$$

$$\text{prob}(l \text{ in info set } n | p^*) = 1 - q;$$

⇒

$$b_R(h | \text{info set } e, p^*) = \frac{\text{prob}(h \text{ in info set } e | p^*)}{\text{prob}(\text{info set } e | p^*)} = 1$$

$$b_R(h | \text{info set } n, p^*) = \frac{\text{prob}(h \text{ in info set } n | p^*)}{\text{prob}(\text{info set } n | p^*)} = 0$$

2) Players strategies sequentially rational?

Receiver:

$$\begin{aligned} \text{info set } e : b_R(h | \text{info set } e, p^*) = 1 &\Rightarrow \\ u_R(p^*, b_R) = 4 + \epsilon > u_R(p_S^*, (b_e, \cdot), b_R) = 2 \end{aligned}$$

$$\begin{aligned} \text{info set } n : b_R(l | \text{info set } n, p^*) = 1 &\Rightarrow \\ u_R(p^*, b_R) = 2 > u_R(p_S^*, (\cdot, a_n), b_R) = 0 \end{aligned}$$

Sender:

info set $h : u_R(p^*) = 3 > u_R((m_n^h, \cdot), p_R^*) = 2$

info set $l : u_R(p^*) = 2 > u_R((\cdot, m_e^l), p_R^*) = 1$ ■

Note: This results holds for all $\epsilon \geq 0$: Even if a university education has no direct impact on the productivity, it gives the better job.

Reason: University degree serves as a signalling device, if non-monetary costs of getting the degree are high enough for the less talented persons so that they do not go for it \implies

it always pays for the students if the study program makes them work hard.