Lecture 5. Static Games of Incomplete Information

Till now: All players know the payoffs of the other players - games of complete information

Now: Players know only their own payoffs, but not that of other players - Games of incomplete information

Games of incomplete information - Bayesian games

Examples:

auctions: Bidders know typically only their own valuation of the good, but not the value other bidders attribute to the good.

Trade negotiations: The seller knows the minimum price at which he is willing to sell, but not the maximum price at which the buyer is willing to buy. The buyer knows the maximum price at which he is willing to buy, but not the minimum price at which the seller is willing to sell.

Oligopolies where firms know their own cost-structure, but not that of competing firms.

1. Cournot competition under Asymmetric Information

Inverse demand for the good:

$$P(Q) = a - Q, \text{ with } Q = q_1 + q_2$$

2 firms produce the good

Firm 1:

$$C_1(q_1) = c_1 q_1$$

Firm 2 can have low or high production costs:

$$C_2^t(q_2^t) = c_{t_2} q_2^{t_2} \text{ with } t_2 = L, H \text{ and } c_L < c_H$$

Firm 2 knows its type, but firm 1 does not. It only knows that 2 is with probability θ high type, and with $1 - \theta$ low type.

Both firms choose simultaneously their quantities - static game

Assume that firm 1 chooses q_1.

Choice of firm 2 depends of course on own type t_2:

$$\max_{q_2^{t_2}} \pi_2^{t_2} = (a - q_1 - q_2^{t_2}) q_2^{t_2} - c_{t_2} q_2^{t_2} \text{ with } t_2 = L, H$$

FOCs:

$$\begin{aligned} 0 &= a - q_1 - 2q_2^H - c_H \\ 0 &= a - q_1 - 2q_2^L - c_L \end{aligned}$$

Reaction functions:

$$\begin{aligned} q_2^H(q_1) &= \frac{a - q_1 - c_H}{2} \\ q_2^L(q_1) &= \frac{a - q_1 - c_L}{2} \end{aligned}$$

Choice of firm 1: Firm 1 does not know type of firm 2, but it knows that 2 types are around, that they make a different choice, and that type H has probability $\theta \Rightarrow$

$$\max_{q_1} \pi_1 = \theta[(a - q_1 - q_2^H)q_1 - c_1 q_1] + (1-\theta)[(a - q_1 - q_2^L)q_1 - c_1 q_1]$$

FOC

$$0 = \theta(a - 2q_1 - q_2^H - c_1) + (1-\theta)(a - 2q_1 - q_2^L - c_1)$$

Reaction function:

$$q_1(q_2^H, q_2^L) = \frac{\theta(a - q_2^H - c_1) + (1-\theta)(a - q_2^L - c_1)}{2}$$

Solving the three reaction functions for the quantities gives the Nash-equilibrium

$$
\begin{aligned}
q_2^{H*} &= \frac{a-2c_H+c_1}{3} + \frac{(1-\theta)}{6}(c_H-c_L) \\
q_2^{L*} &= \frac{a-2c_L+c_1}{3} + \frac{\theta}{6}(c_H-c_L) \\
q_1^{*} &= \frac{a-2c_1+\theta c_H+(1-\theta)c_L}{3}
\end{aligned}
$$

2. Normal Form Game with Incomplete Information - Static Bayesian Game

Idea: Before the game is played, nature chooses a type for each player according to a commonly know probability distribution. Each player is informed about his own type, but not about the actual types of the others. ("Type is private information")

Then each player chooses his action, which may depend on his own type.

$i = 1 ... n$ players

T_i, set of possible types of player i; typical element $t_i \in T_i$

$t = (t_1, t_2, ... t_n)$ type profile

t_{-i} : type profile for all players but player i. (profile of types of the other players). $t_{-i} \in T_{-i}$

p : probability distribution over set of type profiles; $p(t)$ is the commonly known a-priori probability, that nature chooses profile t.

p_i: belief of player i about the types of the others; derived from p

$p_i(t_{-i}\,|t_i)$: belief of player i, that the type profile of the others is t_{-i}, when i is of type t_i.

Since players are rational, beliefs are generated by Bayesian updating:

$$p_i(t_{-i}\,|t_i) = \frac{p(t_{-i}, t_i)}{\sum_{\bar{t}_{-i} \in T_{-i}} p(\bar{t}_{-i}, t_i)}$$

A_i player $i's$ action set (action space); for Bayesian game not the same as strategy set (see below).

u_i : player $i's$ payoff function; payoff depends on combination of actions of all players, and on own type

$$u_i(a_1, ...a_n; t_i)$$

Static Bayesian game $G = \{A_1, ...A_n; T_1, ...T_n; p_1 ...p_n; u_1, ...u_n\}$

Example: Cournot Game with asymmetric information

$A_1 = A_2 = \mathbb{R}_+$

$T_1 = \{t_1\}$; $T_2 = \{L, H\}$

$p_2(t_1 \,|L) = p_2(t_1 \,|H) = 1$;

$p_1(H \,|t_1) = \theta$,
$p_1(L \,|t_1) = (1 - \theta)$

$\pi_1(q_1, q_2, t_1) = (a - q_1 - q_2)q_1 - c_1 q_1$

$\pi_2(q_1, q_2, H) = (a - q_1 - q_2)q_2 - c_H q_2$
$\pi_2(q_1, q_2, L) = (a - q_1 - q_2)q_2 - c_L q_2$

If T_i has one element for every player i, then normal form game with complete information.

Structure is very flexible; it can describe all sort of private information. e.g. if different types of player i differ in what they can do (i.e. differ in the set feasible actions) $\Rightarrow$ utility of a type and an action not feasible to this specific type is $-\infty$.

3. Strategies and the Nash Equilibrium of the Static Bayesian Game

Definition: Take $G = \{A_1, ...A_n; T_1, ... T_n; p_1, ...p_n; u_1, ...u_n\}$. A pure strategy for a player i is a function $s_i(t_i)$ where for each type $t_i \in T_i$, $s_i(t_i) \in A_i$ specifies the feasible action that type t_i would choose if drawn by nature.

Example: Cournot duopoly with asymmetric information

$$\begin{aligned} s_1 &= q_1 \in \mathbb{R}_+ \\ s_2 &= (q_2^H, q_2^L) \in \mathbb{R}_+^2 \end{aligned}$$

Definition: Take $G = \{A_1, ...A_n; T_1, ... T_n; p_1, ...p_n; u_1, ...u_n\}$. The strategy combination $s^* = (s_1^*, s_2^*, ...)$ is a pure strategy Bayesian Nash Equilibrium if for each player i and for each of $i's$ types $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \,|t_i\,) u_i \left[s_1^*(t_1), ..s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..s_n^*(t_n); t_i\right]$$

"No type of no player can increase his expected payoff by choosing a different action."