13. Market Fluctuations I: Scaling, Multiscaling, and their Possible Origins

Thomas Lux and Marcel Ausloos

In this chapter, we provide a survey of research on scaling phenomena in financial data pursued by physicists and compare their methodology and results with the approach of economists dealing with the same topic. We also try to put this work into perspective by discussing in how far it is reconcilable with traditional models in finance (the efficient market hypothesis) or whether it leads to a new viewpoint on market interactions.

13.1 Introduction

In the following we present a selective review of research on the statistical properties of financial data with an emphasis on those features that have also been the focus of recent analyses by physicists. We highlight that many of the observations known as 'stylized facts' (economics terminology) of financial markets can be interpreted as ‘scaling laws’ in the sense of statistical physics. Although the latter term has been virtually unknown in the pertinent economics literature until recently, some ubiquitous statistical findings like the ‘fat tails’ of the distribution of returns and the phenomenon of volatility clustering can, in fact, be cast into the framework of scaling theory. Reviewing work on these empirical characteristics by both economists and physicists, we compare their respective methodology and results. Although we find that, in

Fig. 13.0. Real and synthetic financial ‘time series’: two of the four exhibits are daily records of well-known financial indices The two remaining plots are generated from numerical simulations of two recently proposed models. One of them is a realization of a new type of stochastic process: the multifractal model proposed in [13.114]. In this approach, a multifractal cascade serves as a time transformation which expands and contracts the timescale of a homogeneous incremental Brownian motion. The last graph stems from simulations of the behavioral model proposed in [13.309, 13.130]. Here the price dynamics results from a microscopic model of a large ensemble of interacting agents. The main text identifies the model-generated records and empirical data.
the end, many of the results obtained by both groups are in good qualitative agreement, comparison of the outcome of different test procedures also suggests that one should be cautious in attaching excessive credibility to numerical parameter estimates. We also discuss the implications of scaling in finance for the theoretical modeling of the price formation process. On the one hand, the empirical scaling laws are reconcilable with traditional models of financial markets: since, in principle, we could trace back scaling of prices and returns to similar behavior of the flow of new information, the finding of scaling laws per se does not serve as evidence against perfectly efficient and unbiased information processing (the efficient market hypothesis (EMH)). On the other hand, the experience that scaling laws could often be explained by the working of systems of many interacting units in statistical physics suggests a different approach: from this perspective, some of the salient features of the data might be explained by the interactions of a large ensemble of heterogeneous market participants (the interacting agent view). Some recent models provide first steps into this direction.

13.2 Scaling in the Probability Distribution of Returns

13.2.1 Early Work and Theoretical Background

Research on the statistical properties of financial time series has started 101 yr ago, when Louis Bachelier wrote a Ph.D. Thesis with the title ‘Théorie de la Spéculation’ (reprinted in [13.34], for the first page see Chap. 12, p. 363). In fact, in proposing the normal distribution as a model of price variations, Bachelier seemed to have been the first scientist who formulated a testable hypothesis on the statistical behavior of financial data.

Although it turned out later that the normal distribution can usually be overwhelmingly rejected by explicit statistical tests, due to its familiarity it is still often used in theoretical work as well as by market practitioners. Theoretically, the appeal of the Gaussian stems, of course, in finance like in other fields, from its stability-under-addition property: i.e. the Gaussian distribution is the limit distribution for sums of independent and identically distributed (i.i.d.) random variables (the central limit theorem). This property, in fact, comes into play quite naturally when dealing with financial prices: the most easily available data, price changes on a daily frequency, can be considered as the sums of a multitude of smaller increments on the intra-daily level. If it is reasonable to assume that these high-frequency variations are i.i.d. random variables, then the central limit law should apply to price changes of a lower frequency and it, then, seems natural to postulate that their distribution approaches a Gaussian
shape. But why should we assume that single price changes are governed by random motion? Shouldn’t there be systematic economic factors behind price variations in financial markets? Thus, before looking at empirical results, we will shortly outline the economic reasoning behind the i.i.d. assumption for price changes.

For a long time, the only available theoretical background to the statistical behavior of asset prices has been the so-called efficient market hypothesis (EMH) (cf. [13.48]). It states that, at any point in time, asset prices should reflect the discounted expected stream of earnings from holding the underlying asset. Thereby, the expectations are to be understood in a mathematical sense as the expected future values of stochastic quantities, which are computed using all presently known information about future circumstances.

Although mutatis mutandis the efficient market hypothesis can be applied to other financial markets as well (for example, foreign exchange, precious metal, or future markets), we will concentrate on share markets for ease of exposition. In this case, future earning streams are most easily identified as dividend payments. The price predicted by the EMH should, therefore, be given by:

$$p_t = \sum_{i=1}^{\infty} \delta^i E\left[ d_{t+i}|I_t \right].$$

(13.1)

In (13.1) $p_t$ denotes the price at time $t$, $d_{t+i}$ denotes the dividend paid in a future period $t + i$, and $\delta < 1$ is a discount factor reflecting the fact that the expected payment of one dollar in the future has less value today than immediate availability of the same amount. $E[\cdot]$, finally, is the mathematical expectation operator which is computed conditional on the current information set $I_t$.

We can also express the asset pricing equation in a somewhat different manner if we do not assume that the investor will hold the asset forever, but rather considers to sell it at time $t+1$ in order to consume his receipts or make another investment. Under this perspective, future earnings are composed of the dividend plus the price at $t+1$ which are both uncertain in $t$. Hence, the price should equal:

$$p_t = \delta E\left[ p_{t+1} + d_{t+1}|I_t \right].$$

(13.2)

Iterating (13.2) and repeatedly replacing prices by future dividends, it is, in fact, easy to show that both variants are identical. However, there is a slight difference between (13.1) and (13.2): in order to make sure that the sum on the right hand side of (13.1) converges, we have to assume that the mathematical ‘transversality condition’ $\lim_{i \to \infty} \delta^i E[\cdot|I_t] \to 0$ holds. Although this condition may appear innocuous, a large body of literature has developed about this
and similar conditions in applications in finance [13.51] as well as in other branches of economics [13.72]. In a finance context, its violation allows for price paths which do not conform to the efficient market hypothesis and give rise to so-called rationally expected speculative bubbles\(^1\).

Let us assume that the transversality condition holds and prices can be expressed by (13.1). Then, price changes come about by arrival of some new items of information about future dividends (i.e. an increase of the information set) which changes the rationally computed expectations of future prices and dividends. Of course, such information continuously hits the market: one can, for example, think of such elementary items like weather or climatic events affecting output in certain industries, political changes (like introduction of new taxes on emissions, changes of monetary policy) as well as any kind of firm-specific events (e.g. all types of news about management policies). According to the EMH, the entirety of all these factors would explain the history of price changes: the 'news arrival process' is crucial for and, in fact, governs the characteristics of the distribution of price increments. As 'news' should be events that are independent of the preceding ones, the 'independence' part of the i.i.d. assumption can be immediately justified by the above considerations. If we are prepared to accept the second part as well (identical distribution), we end up with the central limit law for low-frequency price changes as the aggregate of high-frequency ones. Note that under the EMH, the distribution of price changes is, then, a mere reflection of the distribution of arriving news (which, however, is an aggregate of diverse individual events that cannot be quantified). Scaling laws for price changes would, therefore, have to be explained by similar scaling laws for the 'news arrival process'.

13.2.2 Mandelbrot’s Stable Distribution Hypothesis and Alternative Models

During the 1960s and 1970s, the above picture has undergone some changes: first, it has been found that price increments themselves are less suitable for statistical analysis than relative changes. One, therefore, routinely transforms the raw time series of prices into returns: 

\[ r_t = \frac{p_t - p_{t-1}}{p_{t-1}} \approx \ln(p_t) - \ln(p_{t-1}) \]

for statistical analysis. A second and more important modification was brought about by Mandelbrot’s and Fama’s finding that daily returns are decisively

\(^1\)The most elementary example is a price path containing a bubble term \(B_t\) which develops as \(B_{t+1} = B_t \delta\). It is easy to check that this would be in harmony with (13.2), cf. [13.17]. Nevertheless, the price would not be an efficient one just because it incorporates the bubble term \(B_t\). As the literature on rational bubbles has only weak links to the statistical literature we dispense with a more detailed treatment of this research here. In [13.111] it is shown that scaling laws derived from theoretical models of rational bubbles are quite different from empirical ones. In particular, instead of the ubiquitous empirical finding of approximately cubic scaling in the tail regions of the unconditional distribution (cf. Sects. 13.2.3 and 13.2.4 below), rational bubbles give rise to exponents smaller than 1.
nonnormal (cf. [13.47, 13.112]). In fact, standard tests for normality usually reject their null hypothesis at extremely high levels of significance.

Giving up normality, but maintaining stability under aggregation, Mandelbrot [13.112] proposed the Lévy stable distributions as an alternative model. Under this new hypothesis on the distribution of returns, relative price changes would have no finite second and higher moments and would, therefore, obey the generalized central limit law under time aggregation. Besides their desirable stability property, the Lévy distributions are also in harmony with the usual visual appearance of the empirical distribution for returns: they possess more probability mass in their tails and center than the Gaussian (i.e. they are leptokurtotic). Unfortunately, the stable Lévy distributions lack an analytical closed-form solution and can only be described by their characteristic function:

$$\log E(e^{i\omega t}) = \begin{cases} 
    i\delta t - |ct|^\alpha \left[ 1 - i\beta \text{sign}(t) \tan(\pi \alpha_s/2) \right] & \text{if } \alpha_s \neq 0, \\
    i\delta t - |ct| \left[ 1 + i\beta(2/\pi) \text{sign}(t) \log |t| \right] & \text{if } \alpha_s = 1,
\end{cases} \quad (13.3)$$

with parameters $\alpha_s$, $\beta$, $c$, and $\delta$ determining the shape, skewness, width, and location. The distribution function and density of $x$ can only be obtained numerically by evaluating the inverse Fourier transform of the characteristic function (13.3).

Furthermore, at the time of publication of Mandelbrot’s and Fama’s papers, almost no statistical method was known for estimating the parameters and testing goodness-of-fit of the Lévy distributions. Nevertheless, the argument of convergence of aggregate returns towards one of the stable laws appeared so convincing, that a number of researchers accepted indication of $\alpha_s < 2$ as evidence in favor of Lévy stable distributions without further testing of fit. Examples in the literature include [13.35, 13.138, 13.139, 13.151]. At the same time, Mandelbrot’s hypothesis was questioned by others on the basis of cleverly designed tests for stability-under-addition itself. These tests often produced nonstationary results for the characteristic exponent $\alpha_s$ at different levels of time aggregation, which is in contradiction to the assumed stability of the distributions (cf. [13.55, 13.70, 13.78, 13.153]). Similarly, analysis of the behavior of higher moments produced evidence against the Lévy model in that the divergence proceeded more slowly than expected [13.89].

For more than 25 yr, these two conflicting strands of research coexisted in the literature and the issue of the appropriate distributional assumptions remained basically undecided. However, from a practical point of view, a number of alternative, more easily tractable statistical models have also been proposed,

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2 From his recent publications, it seems that Mandelbrot has also converged to acceptance of finite second moments, cf. [13.30].
although they lack a strong theoretical foundation. From the wealth of papers that appeared over the decades, only a few are mentioned here emphasizing the broad spectrum of distributions that have been proposed:

- mixtures of normal distributions with different means or variances [13.87, 13.152].
- subordinated processes, most notably Clark’s model [13.30] of a log-normal subordinator for the variance of a Gaussian incremental distribution. Interestingly, Clark used volume data as a measure of trading intensity in order to get a fit of the log-normal.
- compound diffusion-jump processes with the jump component reflecting ‘extraordinary’ events [13.54].
- the Student’s t-distribution as a fat tailed distribution with finite higher moments [13.18].
- Tuckey’s ‘$g \times h$ distributions’ (transformations of the normal that allow for skewness and leptokurtosis [13.121].
- hyperbolic distributions [13.43].

13.2.3 The Contribution of the Statistical Theory of Extremes

This huge variety of distributional forms makes it hard to find universal laws in the statistical behavior of financial prices. Starting in the early 1990s, a new type of analysis has appeared in empirical finance, that, in fact, highlights typical features by abstracting from a specific distributional shape. This strand of literature took up a proposal by DuMouchel [13.42] to concentrate on the behavior of the tails instead of trying to fit the entire distribution. The theoretical background to this research program is provided by statistical extreme value theory (laid out, for example, in the following recent textbooks: [13.13,13.133]).

One of the basic results of the theory of extremes is typology of the limiting distributions for maxima (or minima) from i.i.d. random variables with continuous distributions. Namely, denoting by $M = \max(x_1, x_2, \ldots, x_n)$ the maximum of a sample of observations $\{x_i\}$, it can be shown that after appropriate change of scale and location the limiting distribution of $M$ belongs to one of only three classes of distribution functions. Expressed more formally, the limiting distribution of the normalized maximum, $P[a_n M + b_n \leq x]$, with $a_n$ and $b_n$ denoting normalizing constants, converges to one of the following types of generalized extreme value (GEV) distributions:
\[ G_{1,\alpha}(x) = \begin{cases} 0 & x \leq 0, \\ \exp\left(-x^{-\alpha}\right) & x > 0, \end{cases} \quad (13.4) \]
\[ G_{2,\alpha}(x) = \begin{cases} \exp\left(-(-x)^\alpha\right) & x \leq 0, \\ 1 & x > 0, \end{cases} \quad (13.5) \]
\[ G_3(x) = \exp\left(-e^{-x}\right) \quad x \in \mathbb{R}. \quad (13.6) \]

As an alternative to (13.4) to (13.6) one also encounters the so-called von Mises representation of the GEV distributions which provides a unified framework for all three cases:

\[ G_\gamma(x) = \exp\left(-\left(1 + \gamma x/\sigma\right)^{-1/\gamma}\right). \quad (13.7) \]

In this formalization, \( \sigma > 0 \) is a scale parameter and the three elementary types of extremal behavior are characterized by \( \gamma > 0, \gamma < 0, \) and the limit in the case \( \gamma \to 0. \) For type 1 (2), the shape parameters of both representations (1) and (2) are related to each other by the following identities: \( \gamma = 1/\alpha (\gamma = -1/\alpha). \)

From this classification of extrema, a similar classification of the behavior in the distribution's outer parts can be derived. Namely, denoting the probabilities \( \text{Prob}[X_i \leq x] \equiv W \), it follows directly from the classification of extremes in (13.4)-(13.6) that if the maximum of a distribution follows a GEV of type \( i \) \( (i = 1, 2, 3) \) then the upper tail of the distribution is close to

\[ W_{1,\alpha} = 1 - x^{-\alpha}, \quad x \geq 1, \quad (13.8) \]
\[ W_{2,\alpha} = 1 - (-x)^\alpha, \quad -1 \leq x \leq 0, \quad (13.9) \]
\[ W_3 = 1 - \exp(-x), \quad x \geq 0. \quad (13.10) \]

i.e. a so-called generalized Pareto distribution (GPD) having the same tail shape parameter \( \alpha \) (cf. [13,133], Chap. 5).

As with the GEVs, one can also integrate the three laws (13.8)-(13.10) into one unifying representation:

\[ W_\gamma = 1 - (1 + \gamma x/\sigma)^{-1/\gamma}, \quad (13.11) \]

The three elementary types of tail behavior can be described as hyperbolic decline (first case), distributions with finite endpoints (case 2), and exponential decline (case 3). They are characterized by \( \gamma > 0, \gamma < 0, \) and the limit in the \( \gamma \to 0 \) case. For type 1 (2), the shape parameters of both representations (13.8)-(13.10) and (13.11) are again related to each other by the following identities: \( \gamma = 1/\alpha (\gamma = -1/\alpha). \)
A variety of estimators has been developed for the ‘tail index’ $\alpha$ or $\gamma$. Estimation of this quantity for the tail shape alone is favorable over an overall fit of the empirical distribution for a number of reasons: first, it gives an indication of the type of behavior in the outer parts and, therefore, may allow us to exclude a number of candidate processes from the outset (e.g., an estimate of $\gamma$ significantly different from zero would simultaneously imply rejection of the normal, mixtures of normals as well as diffusion-jump processes, as they all lack the required behavior of large returns). Second, in financial applications, one is often more interested in extreme realizations (crash risks!) than in the exact shape of minor fluctuations. As in any fit of the overall distribution, the extremes only constitute a small part of the sample, their ‘influence’ on the final estimation result is almost negligible. Furthermore, the choice of a particular overall model already restricts the results that can be found for the tail behavior (for example, choice of the Lévy distributions always implies that tails are assumed to be extremely heavy and restricts the tail index to the region $\alpha < 2$).

The majority of studies in the literature applied maximum likelihood estimators for the tail index $\alpha$ or $\gamma$. Because of its simplicity the ‘Hill tail index estimator’ [13.74] has become the standard tool in most studies of tail behavior of economic data. The Hill estimate $\hat{\gamma}_H$ is obtained by maximization of the likelihood of the relevant tail function conditional on the chosen size of the ‘tail’. It is computed as:

$$
\hat{\gamma}_H = \left( \hat{\alpha}_H \right)^{-1} = \frac{1}{k} \sum_{i=1}^{k} \left[ \log x_{(n-i+1)} - \log x_{(n-k)} \right].
$$

(13.12)

In (13.12), sample elements are put in descending order: $x_{(n)} \geq x_{(n-1)} \geq \ldots \geq x_{(n-k)} \geq \ldots \geq x_{(1)}$ with $k$ the number of observations located in the ‘tail’.

The results from a large body of research in this vein turned out to yield astonishingly uniform behavior for data from different markets in that one usually finds a tail index (exponent of the cumulative distribution) in the range 2.5 to about 4. Examples include [13.81, 13.86, 13.99, 13.100, 13.102]. Surveys of this literature can be found in [13.126, 13.162].

Later on, refined statistical methods with data-dependent choice of the tail region (i.e. the number $k$ of observations that follow the approximate tail distribution) confirmed these findings [13.37, 13.107, 13.108]. Similarly, analysis of the tails of high-frequency, intra-daily data also led to results in accordance with those for daily data. This squares well with the theoretically expected invariance of tail behavior at different levels of time aggregation [13.1, 13.124]. It is worth emphasizing that this invariance holds for processes with $\alpha > 2$ despite convergence of the overall distribution to the Gaussian. Therefore, the
laws governing the tails will become less and less ‘visible’ under aggregation and will only show up in the extreme outer parts of extremely large datasets. As the financial data of the formerly prevalent daily frequency are only available in limited numbers, the increased availability of high-frequency data appears particularly valuable. In fact, with intra-daily records, one not only enjoys a tremendous increase of the sheer number of data points to be used, but also has available data at an ‘earlier’ stage of time aggregation, so that one could hope for an extended range of validity of the pertinent tail laws.

13.2.4 Recent Contributions by Physicists

Physicists would, of course, interpret the findings of large returns \( r_t \) following a relationship like \( \text{Prob}(r_t > x) \sim x^{-\alpha} \) implied by the tails obeying \((13.8)\) as the typical appearance of a power-law. However, one should note the difference between the statistical arguments outlined above and the concept of scaling laws in physics: while scaling laws should ideally extend over a large range of observations (typically several decades), the arguments from extreme value theory only imply that the most extreme part of the distribution will asymptotically converge to a limiting power-law. For a particular dataset, the region where this power-law can be ‘observed’ may be rather small and the analysis of a larger portion of the data with similar tools may already give different results.

The developments in empirical finance sketched above are mirrored to some extent in the recent work by physicists on distributional aspects of financial data. To our knowledge, the first contribution in this vein is an article by Mantegna \([13.116]\) who estimates the parameters of the stable laws (cf. \((13.3)\)) for various sectoral indices of the Milan stock exchange. His results are comparable to those obtained by economists in similar studies. In later work, Mantegna and Stanley \([13.117]\) as well as Cont et al. \([13.33]\) remarked that Lévy-type behavior does not extend to the most extreme parts of the distribution and replaced the original Lévy hypothesis by that of a truncated Lévy process whose tails fall off exponentially after some threshold value. A few years later, a paper from the same group \([13.63]\) arrived at the conclusion that tails are rather governed by a cubic power-law, that is, their tail index estimate is close to three. Thus, it seems that researchers from economics and physics have completely converged on this issue.

As an interesting side aspect of the above discussion, it appears that the central parts of the empirical distributions of returns are closer to the shape of the stable laws than the extremal region. This can be seen in Mantegna and Stanley \([13.117]\) who have shown that the estimated Lévy law provides a good fit to high-frequency returns of the S&P 500 index up to some high
Table 13.1. Scaling of extreme returns in various financial time series. Asymptotic 95% confidence intervals of the Hill estimator [33,74] are given in parentheses. The time intervals and number of observations are: DAX: 10/59-12/98 (n = 9818), NYCI (New York Stock Exchange Composite Index): 01/66-12/98 (n = 8308), USD-DEM: 01/74-12/98 (n = 6140), Gold price: 01/78-12/98 (n = 5140). Note that all estimates point to a scaling exponent around three. Furthermore, the confidence intervals of the Hill estimates from the 5% tail as well as from the usually smaller ‘optimal’ tail allow rejection of the stable law hypothesis $\alpha < 2$. ‘Optimal’ tail sizes are computed with the stopping algorithm of [33,39].

<table>
<thead>
<tr>
<th>Data</th>
<th>Regression estimate</th>
<th>Hill 5%</th>
<th>Hill estimate based on optimal tail size</th>
</tr>
</thead>
<tbody>
<tr>
<td>DAX</td>
<td>2.92</td>
<td>3.01</td>
<td>(2.74, 3.27)</td>
</tr>
<tr>
<td>NYCI</td>
<td>2.31</td>
<td>3.58</td>
<td>(3.24, 3.93)</td>
</tr>
<tr>
<td>USD-DEM</td>
<td>3.35</td>
<td>3.60</td>
<td>(3.28, 4.10)</td>
</tr>
<tr>
<td>Gold</td>
<td>2.67</td>
<td>2.51</td>
<td>(2.20, 2.81)</td>
</tr>
</tbody>
</table>

threshold value (see Chap. 12, Fig. 12.2). Similarly, Groenendijk et al. [13,67] highlight the fact that certain estimation methods for the parameters of stable distributions based on the interquartile range give quite uniform results across assets. Again, the likely reason is that the implicit restriction to the inner parts of the distribution with some methods makes empirical records look closer to the stable laws than investigation of the tails. Note also that similar conclusions are confirmed by tail index estimation itself. If we were to consider a tail size of 30% and more appropriate for maximum likelihood estimation of $\alpha$, we would end up with a tail index in the range 1 to 2, that is in the basin of attraction of the stable laws. However, data-driven choice of the tail fraction to be used in this type of computation tells us that we have to go further out into the tail in order to get a reliable estimate of $\alpha$ - with the usual finding that $\alpha$ is around 3–4 as outlined above.

Figure 13.1 and Table 13.1 provide illustrations of the typical scaling behavior of the tails of the unconditional distribution of financial returns. In Fig. 13.1 we exhibit results obtained with both daily and intra-daily recordings of the German share price index DAX. Figure 13.1a shows that the cumulative distributions of both time series exhibit clear deviations from the exponential decline of a normal distribution and are better characterized by hyperbolic power-law

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3 At the end, the Lévy type appearance of the distribution's center may have a quite mundane explanation: on the highest frequency, the peak at the center is simply due to a very large number of observations exactly equal to zero (no trade). Furthermore, at this frequency, the distribution is discontinuous because the minimum tick size is finite in most markets. Temporal aggregation, then, amounts to smoothing over this initially kinked peak at zero which may produce the artificial impression of a smooth shape (personal communication by P. Gopalkrishnan).
Fig. 13.1. Scaling of extreme returns from a typical financial time series (returns of the German share price index DAX). (a) The complement of the cumulative distribution function at different levels of time aggregation. The data under consideration are daily observations over the years 1990 to 1998 and minute-to-minute returns during opening hours from 1998 to 1995. For both daily and intra-daily frequencies, we observe approximate power-law behavior of the tails. Performing a least squares fit of the most extreme 20% of the data, we find exponents 2.92 (daily data) and 3.06 (minute-to-minute returns). At lower frequencies (e.g., returns over 20 days) a crossover to the normal distribution (with mean zero and unit variance) is observed. In order to enhance comparability, returns have been rescaled by dividing by the sample standard deviation at all levels of time aggregation. (b) Exhibits the results of maximum likelihood estimation of the tail index for daily DAX data. When varying the number of data considered to make up the ‘tail’, a plateau with $\alpha \approx 3.1$ is found and a monotonic decline of the estimate thereafter. Statistical methods for selecting the optimal tail size usually give numbers in the vicinity of the plateau. The bottom panel shows the time development of both the third moment (c), and fourth moment (d). Apparent convergence of the third moment and seeming nonconvergence of the fourth moment square well with our estimates of the tail index.

behavior. Since one usually finds only slight differences between the left and right tails, we have merged both sides by using absolute values of returns. Log-log regression estimation of the shape of the distribution in the extremal region yields very similar results with estimates around three for both time series. However, at a higher level of time aggregation (exemplified by returns over 20 days), the visibility of power-law scaling breaks down and the distribution approaches the normal. Maximum likelihood estimates using the Hill
technique [13.74] are exhibited in Fig. 13.1b. Here we do not restrict ourselves
to only one estimate, but show the typical variation of the tail indices with
increasing number of tail elements, $k$. Upon inspection, one detects a plateau
of the estimate with $\hat{\alpha}_H \approx 3.1$ for tail sizes between about 5 and 10% of the
data, and monotonic decline thereafter. Usually, refined methods for endoge-
nous selection of $k$ tend to choose estimates around this almost stable region.
In this case, the outcome is in almost perfect agreement with that obtained
by log regression. The validity of the results is further confirmed by inspection
of the development of the third and fourth sample moments (computed with
daily data): while the third moment appears to converge with sample size, for
the fourth moment convergence appears doubtful and it seems likely that this
quantity suffers from nonstationarity (Fig. 13.1c-13.1d).

Table 13.1 shows numerical results obtained with various important financial time series: besides the DAX, we also consider the New York Stock Ex-
change Composite Index (NYCI), the USD/DEM exchange rate, and the price
of gold\textsuperscript{4}. Although we find some variation across assets, the picture confirms
the overall impression from the literature with estimates hovering between 2.5
and 4. Comparing regression estimates with two sets of Hill estimates computed from both the 5 percent tails and an endogenous selection method for
$k$ [13.39], we see some fluctuations, but hardly any systematic differences. It
is also interesting to note that explicit hypothesis testing based on the 95%
confidence intervals exhibited for the Hill estimates allows us to reject the Lévy hypothesis $\alpha < 2$ in all cases.

### 13.3 Temporal Dependence

#### 13.3.1 Dependence in Raw Returns: Theoretical Background and Empirical Evidence

As pointed out above, the efficient market hypothesis implies that price changes
are caused by newly arriving information which should itself follow some sort
of stochastic process. Denoting price increments by $\varepsilon_t$ and maintaining the
assumption that increments are i.i.d. stochastic variables with mean zero, the
price itself would, then, follow a random walk:

$$p_{t+1} = p_t + \varepsilon_t . \quad (13.13)$$

This implies that the best forecast of tomorrow’s price is the price observed
today:

$$E[p_{t+1} | I_t] = p_t . \quad (13.14)$$

\textsuperscript{4} The bottom panels of Fig. fig:lux show parts of the DAX (left-hand side) and USD/DEM time series (right-hand side).
Strictly speaking, however, (13.1) and (13.2) do not imply that prices themselves should follow a random walk. Rather, one can show that this property holds for a closely related quantity, the value of an initial investment whose returns are continuously reinvested in the same asset (cf. [13.91]). Nevertheless, in empirical research, the efficient market hypothesis has often been identified with the random walk character of prices or their logs: \( \ln(p_{t+1}) = \ln(p_t) + \varepsilon_t \).

The latter form, of course, is in harmony with returns rather than price increments following an i.i.d. stochastic process.

As certain statistical features of empirical data (to be dealt with in detail below) are not consistent with the random walk model, it has later on mostly been replaced in the financial literature by the wider concept of a martingale process. This concept implies that prices (or their logs) are still assumed to follow a process of the type of (13.13) but with the only restriction imposed on increments that \( E[\varepsilon_t] = 0, \forall t \). This allows for nonidentical distributions of the increments as well as for many forms of dependence in higher moments and is, nevertheless, in harmony with the EMH since price changes remain unpredictable under the martingale hypothesis (cf. [13.27]).

A wealth of research has tried to find evidence in favor of or against the hypothesis in (13.13). Supporting evidence comes from so-called unit-root tests: if one embeds (13.13) into a time series model of the more general type \( p_{t+1} = \rho p_t + \varepsilon_t \) one can perform a test of the hypothesis \( \rho = 1 \) (unit-root test). This is equivalent to a standard \( t \)-test of a hypothesized coefficient in a regression, but since the value \( \rho = 1 \) is at the boundary between the stationary and nonstationary cases, the distribution of the test statistic is nonstandard (cf. [13.40]). Results of this and a number of related tests are quite uniform in that one is usually unable to reject the above null hypothesis for financial prices (or their logs).

On the other hand, i.i.d.-ness of (relative) price changes would imply absence of any significant autocorrelation in the returns series. In a strict sense, this implication of the random walk model is often violated in that one finds significant correlation over one or two lags for data on a daily level (cf. [13.27]), and over a couple of minutes for intra-daily data (cf. [13.119], Chap. 7). However, the economic interpretation is that these slightly significant correlations can be attributed to market frictions and market imperfections especially with thinly traded assets and, therefore, may not really be considered as a rejection of the underlying economic idea of the EMH\(^5\). More refined tests try to show, for example, that prices systematically tend to overreact to news [13.38] or that prices are excessively volatile compared with fundamental factors [13.137]. These as well as related issues are still in the center of current debates. From

\(^5\) For indices, some small positive correlation can even come about in an artificial way from the aggregation of stocks itself.
the literature available so far, evidence seems to be in favor of the overreaction and excess volatility hypotheses (cf. [13.52, 13.163]).

Another possible avenue for attacking the unit-root character of prices is to demonstrate prevalence of hidden nonlinear structure. Again, this has been a very active area of research over the last decade, so that we cannot give an exhaustive overview of relevant work here. Introduction of concepts from chaos theory started with Scheinkman and LeBaron [13.135] and Frank and Stengos [13.53]. While the first vintage of investigations seemed somewhat supportive of chaotic attractors, these results have been questioned by later authors (cf. [13.60, 13.134]). After about one decade of research in this vein, it seems that the search for a low-dimensional attractor in financial data has practically been given up and that the majority view among economists is that price dynamics are more complicated than dynamics arising from standard models of low-dimensional chaotic behavior (for example, the Lorenz attractor).

On the other hand, some newly developed explicit tests of the i.i.d. hypothesis (with the so-called Brock-Dechert-Scheinkman (BDS) test the most popular variant among economists) routinely reject i.i.d.-ness of returns (cf. [13.24]). The results of this literature (reviewed in [13.11]) clearly speak against (log) prices following a random walk. However, as the source of rejection of the null hypothesis could be temporal dependence in variances and higher moments, these results can not necessarily be interpreted as evidence against the more general martingale hypotheses.

Still different, but somewhat related attacks on the pure stochastic nature of returns are based on the successes of certain trading strategies. Note that even with the martingale model, no useful information about the direction of future price changes can be inferred from the past price history. Hence, not only all those strategies that are known under the heading of ‘chartism’ but also refined time series techniques like neural networks are bound to remain useless and should not have any advantage over the simple ‘buy-and-hold’ device. Put the other way round, demonstration of excess returns earned through chartist techniques would point towards violation of the martingale assumption. Again, a large number of relevant studies exist on this issue and we shall confine ourselves to mentioning only a few typical examples: Brock et al. [13.25] show that popular chartist techniques like moving average and resistance lines would have earned small excess returns when used over very long horizons, while Caginalp and Laurent [13.26] demonstrate the astonishing predictive capability of ‘candlestick’ techniques. It has often been noticed that tests for the profitability of trading rules (as well as tests of various ‘anomalies’ of price records) do suffer from a data mining bias which emerges from hundreds of practitioners

\footnote{[13.7, 13.157, 13.158] discuss relationships between moving average techniques and certain concepts from statistical physics.}
and academic researchers looking for conspicuous properties and performing hypothesis tests on one and the same dataset. As is well known from introductory statistics, such ‘data snooping’ should be avoided, as seemingly significant results will eventually emerge by chance with repeated tests. In fact, the recent development of statistical methods to account for data snooping biases allowed a reconsideration of the Brock et al. [13.25] analysis of trading rules with the result that in an out-of-sample test excess returns now turned out to be insignificant [13.145]. The statistical test design is a bootstrap algorithm from the whole universe of similar trading rules (a total of 7846 trading rules).

In a similar vein, Sullivan et al. [13.146] show that the seemingly ubiquitous ‘calendar effects’ (abnormal returns related to the day of the week, week of the month etc.) do not survive a correction for data mining. Hence, both earlier results on the profitability of trading rules and the appearance of ‘anomalies’ have to be interpreted with care.

Interestingly, besides the trading methods developed by market practitioners, recent literature has also investigated more refined mathematical methods for pattern recognition. As an example, we mention Neely et al. [13.125] who show that exploitable patterns can be detected in foreign exchange rates by genetic programming (for an introduction to genetic programming cf. [13.88]).

As a summary of our tour d’horizon on the search for dependencies in raw returns, it turns out form a large body of diverse approaches that on a first view and with standard statistical techniques prices look like realizations of a martingale process and, therefore, seem essentially unpredictable. At a closer look and with more refined statistical instruments, however, one can detect a number of slight deviations from pure randomness, so that one might conclude prices are only approximately efficient (cf. [13.49])\(^7\). The deviations from i.i.d.-ness that are statistically obvious appear, however, to be caused mainly by temporal structure in volatility (second moments) and are hardly noticeable in first moments.

### 13.3.2 Dependence in Squared and Absolute Returns

Our claim of the almost complete absence of dependence in raw returns is illustrated in the lower left hand panel of Fig.13.2: autocorrelations of the DAX returns, in fact, do exceed the 95% bounds for the first two lags but for lags larger than two days the autocorrelations are not significantly different from zero any more. However, the picture changes when considering simple transformations of the original data. As also shown in Fig.13.2 (middle and

\(^7\)A similar view is also expressed in Zhang [13.164]. However, it is not clear whether his supporting evidence for inefficiencies, i.e. sign patterns of returns, is statistically significant and, if so, whether significance would survive on condition that a correction for data mining is exerted. In any case, it should be noted that these patterns have been known for a long time in the economics literature, e.g. [13.77]. The paper by Caginalp and Laurent [13.26] provides a more rigorous statistical analysis of sign patterns.
upper left hand panels) for squares and absolute values of returns temporal independence is strongly rejected. On the contrary, we find significant positive autocorrelation coefficients over an extended time horizon. As absolute and squared returns only preserve the scale and neglect the direction of the increments, we can interpret both transformations as simple measures of the scale
of fluctuations, i.e. volatility. The above findings, then, imply that volatility is strongly correlated over time. Hence, expected volatility in the next period is the higher the more volatile today’s market is. Visually, we observe typical alternations between turbulent and tranquil episodes in the data (volatility clustering).

Economists have developed a class of time series models which captures this ubiquitous feature of financial data. This so-called GARCH (generalized autoregressive conditional heteroscedasticity) model assumes that returns are drawn from a normal distribution whose variance follows an autoregressive process including dependence on past squared increments [13.19, 13.44].

In its general form, with an arbitrary number of lags included, this model reads:

\[ r_t = h_t \varepsilon_t , \]
\[ h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} , \quad \varepsilon_t \sim N(0, 1) , \]
\[ \alpha_0 > 0, \quad \alpha_i, \beta_j \geq 0 . \]  

Using statistical information criteria in order to discriminate between different variants of the GARCH\((p, q)\) model, it mostly turns out that the GARCH\((1, 1)\) model is not outperformed by more complicated versions (cf. [13.15, 13.20]). It, therefore, turns out to be sufficient in most cases to include only last period’s squared return \(r_{t-1}^2\) and last period’s variance \(h_{t-1}\) in the difference equation. Empirical estimation yields results that are also relatively constant across markets and time horizons. Interestingly, the sum of the coefficient \(\alpha_1 + \beta_1\)
comes usually close to one, that is, it approaches the threshold where the process becomes nonstationary\(^8\). Refinements of GARCH models have been a very active research field over the last two decades. The surveys by Bollerslev et al. [13.20] and Bera and Higgins [13.15] give an overview of wealth of the variations of the ARCH theme alternative specifications of the dependence structure or alternative distributions of the increments (e.g. Student’s t-test, stable Lévy increments).

In the light of Sect. 13.2, it is interesting to notice that despite its normal incremental distribution the GARCH model leads to power-law tails [13.68] which is in harmony with empirical evidence. However, it has also often been found that the GARCH variance dynamics is unable to capture all of the deviations from Gaussian behavior in the data, as the residuals from an estimated GARCH model are often still non-Gaussian and leptokurtotic (cf. [13.126]). Furthermore, as has been shown by [13.149], the theoretical tail indices calculated for the estimated theoretical GARCH processes are often not identical to those directly estimated from the data.

An even more important disadvantage of the GARCH class is that the decay of the autocorrelations is exponential, i.e. it covers only short-term dependence\(^9\). Typical plots of the empirical ACF in Fig. 13.2 are, however, quite suggestive of hyperbolic decline rather than exponential decline.

### 13.3.3 Long-Term Dependence

Hyperbolic decline of the autocorrelation function is a defining property of stochastic processes exhibiting long memory. Well-known examples of long-memory processes are fractional Brownian motion (fBm, cf. [13.115]) which can be generated as a moving average from standard Brownian increments \(dB(t)\):

\[
x(t) = \int_{-\infty}^{t} (t - s)^{H-1/2} dB(s),
\]

(13.16)

or fractionally integrated autoregressive moving average (ARFIMA) models [13.64]:

\[
\Phi (\mathfrak{B})(1 - \mathfrak{B})^d x(t) = \Theta (\mathfrak{B}) \varepsilon(t),
\]

(13.17)

with \(\mathfrak{B}\) the backward shift operator defined by \(\mathfrak{B} x(t) = x(t - 1)\), \(\Phi (\mathfrak{B})\) and \(\Theta (B)\) the autoregressive and moving-average polynomials and \(\varepsilon(t)\) a noise term following the normal distribution. It has been shown that both the decay of the autocorrelation function and the spectral density function behave identically.

\(^8\) GARCH models are usually estimated by constraint maximum likelihood which constrains the sum of the coefficients to be smaller than 1.

\(^9\) [13.139] also show that empirical autocorrelation functions are usually outside the 95% confidence bands obtained with GARCH processes. Note, however, that some recent developments allow for long-term dependence in an ARCH framework, cf. the fractionally integrated GARCH model (FIGARCH) [13.8].
for both processes with a relationship \( d = H - 0.5 \) of the crucial long memory parameters \( H \) (known as the Hurst exponent, cf. [13.115] and \( d \) (the degree of fractional differencing, cf. [13.58]). As the temporal development of many quantities of interest can be characterized solely by the exponents \( H \) or \( d \), long memory processes are also often encountered under the label of self-similar stochastic processes. Positive autocorrelation (persistence) is obtained with \( H > 1/2 \) or \( d > 0 \), while the case \( H < 0.5 \) (\( d < 0 \)) gives rise to antipersistent (self-avoiding) processes.

Convincing empirical evidence for long memory properties in squared and absolute returns has been reported by a number of papers both from economists and physicists over the last couple of years. In economics, this property seems to have been recognized first by Ding et al. [13.41]. Their findings have been confirmed in a number of other studies recently [13.36,13.103,13.122]. The consensus now is that this feature appears in virtually all financial prices [13.98]. More or less independently, physicists also reported findings of long-term dependence in financial data, e.g. [13.57,13.96,13.154,13.161].

Here again, it is puzzling to see that while the results are almost identical, the physicists' research has been carried out without knowledge of the relevant literature in economics. One of the reasons for the apparent lack of contact and knowledge of parallel studies besides different cultural background and the lack of common meetings (until recently) may be the different preferences concerning statistical methods and the different attitudes towards drawing inferences from statistical results. While most economists prefer methods that allow explicit hypothesis testing, many physicists apparently have a preference for graphical tools of inference. In testing for long-term dependence, the most popular techniques among physicists are the rescaled range (R/S) method (see Chap. 8) or the more robust detrended fluctuation analysis (DFA) (see Chaps. 5, and 7-8 for examples) proposed by [13.128]. The recent papers by economists, on the other hand, mostly use the Geweke/Porter-Hudak [13.58] variant of periodogram regression (for which asymptotic results on the distribution of the test statistic are available) and Lo's [13.97] method which transforms the R/S methodology into an explicit test of the null-hypothesis of short vs. long-term dependence.

Figure 13.2 illustrates the typical outcome of some of the most popular methods (the detrended fluctuation analysis and Geweke/Porter-Hudak methods) for the German share price index DAX. DFA works as follows: consider the cumulative sum of a time series: \( y_t = \sum_{i=1}^{t} x_i \) and divide the resulting sequence \( \{y_t\} \) into \( T/l \) nonoverlapping boxes of length \( l \). Computing the local trend within each box as the result of a least-squares regression the average standard deviation of the detrended walk over the \( T/l \) blocks, \( \bar{F}(l) \), is expected to scale with \( l \) as:
where $H$ again is the Hurst exponent. Equation (13.18) allows us to estimate $H$ from a linear regression in a log-log plot. The DFA introduced here is referred to as DFA1 in Chap. 5, where also the other variants of the detrended fluctuation analysis are being discussed in detail. In Sect. 5.4.2, the fluctuation exponent was denoted by $\alpha$. For long range correlated sequences, the autocorrelation function $C(l)$ decays by a power law according to (5.1), with an exponent $\gamma$ that is related to $H$ by $H = 1 - \gamma/2$.

The second method, in contrast, estimates $d$ in (13.17) from a linear regression of the log-periodogram $\ln\{I(\lambda_j)\}$ on transformations of small Fourier frequencies $\ln\{4\sin^2(\lambda_j/2)\}$ which for a fractionally differenced ARMA process should obey:

$$E[\ln\{I(\lambda_j)\}] = c - d \ln\{4\sin^2(\lambda_j/2)\}. \quad (13.19)$$

Economists usually consider it an advantage that knowledge on the asymptotic distribution of the estimator $\hat{d}$ from (13.19) is available while for the R/S technique despite its long usage no such results could be derived so far.

Evidence in favor of long memory in both squared and absolute DAX returns in Fig. 13.2 is confirmed by similar outcomes for other assets as reported in Table 13.2. Together with the large body of available literature on this issue, they confirm that long-term dependence in squared and absolute returns are an ubiquitous feature of financial data. We therefore have another scaling law here: hyperbolic decline of the autocorrelation function of these measures of volatility.

As also shown in Fig. 13.2, raw returns themselves do not need to be characterized by long-term dependence. Given the absence of significant autocorrelation in the raw data, this result may not come as a surprise (although one could imagine appropriate mixtures of, say, negative short-term dependence with positive long-term dependence that could obscure the underlying dependence structure leading to apparently insignificant correlations). Nevertheless, a lot of research has been devoted to testing for long-term dependence in raw returns, so that a word on this issue is in order here. Quite some time ago, a number of papers in finance have applied rescaled range analysis to raw returns and have reported positive evidence for a self-similarity parameter $H > 0.5$ [13.21, 13.66, 13.83]. However, later reconsiderations of this issue with other techniques (e.g. [13.61, 13.95, 13.97]) could not find convincing evidence in favor of such behavior\(^\text{10}\). From this literature, a certain majority opinion

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\(^{10}\) The recent paper by Taqqu and Teverovsky [13.149], on the other hand, reconsiders the S&P 500 data used by Lo [13.97] and reports some support in favor of small degrees of long-term dependence.
seems to have emerged among economists that the long memory property is confined to power transformations of returns but appears to be absent in the raw data. This is also in harmony with economic intuition as even slight deviations from $H = 0.5$ for returns could be easily exploited by appropriate trading strategies [13.6, 13.75] and would, of course, strongly violate the martingale hypothesis\textsuperscript{11}.

The conclusion of early findings of $H > 0.5$ being due to the upward bias of the $R/S$ technique for processes with self-similarity parameter around 0.5 is also confirmed by recent Monte Carlo studies of the performance of various estimators [13.129, 13.148, 13.150] all found the $R/S$ technique to have larger mean squared error than alternative techniques, so that $H$ estimates in the range [0.4, 0.6] as reported in earlier studies could arise quite easily from sampling fluctuations. Particular doubts on the relevance of some Hurst exponent estimates above 0.5 are raised by Pilgram and Kaplan [13.129] who report that they found the 'typical value' $H = 0.75$ for simulated time series of synthetic $1/f^\sigma$ noises with true $H$ in the range 0.25 to 0.75!

On the other hand, some (negative) deviations from $H = 0.5$ that have recently been reported may be easily explained from an economic point of view by regulations imposed on the market by monetary authorities. As an interesting example, Vandewalle and Ausloos [13.154, 13.155] report estimates of $H$ for a variety of exchange rates (their results are obtained from DFA analysis of multifractality, see below). Many of them, in particular, those of members of the European Monetary System (EMS) of the 1980s and 1990s, turn out to yield $H < 0.5$. However, within the EMS, the obligations of central banks to keep fluctuations within the band creates the necessity of interventions if the margin is approached. As essentially the exchange rate band acts as a reflecting boundary (either explicitly or through the expectations of agents), the managed floating of European currencies contains an element of antipersistence, and $H < 0.5$ could be seen as the immediate statistical consequence of this mechanism.

\section*{13.4 Multiscaling, Multifractality, and Turbulence in Financial Markets}

Absence of long memory in raw returns and prevalence of long-term dependence in squared and absolute returns can be put into a broader context by considering a continuum of power transformations of the raw data. The first paper to look at power transformations other than $q = 1$ and $q = 2$ (abso-

\footnote{Volatility clustering and long-term dependence in second moments are, however, still consistent with the martingale model and the EMH. Given these empirical regularities, the EMH, however, would imply that news come in clusters and with long-term dependence in their scale.}
lute and squared returns), was [13.41]. For a long series of the daily S&P 500 index variations (ranging from 1929 to 1991) they showed that the strongest correlation is obtained around $q = 1$ (absolute returns). This finding has later on been confirmed for other financial data as well (e.g. [13.122]). For physicists, variation of the scaling exponent for various powers of the original data is reminiscent of multifractal or multi-affine behavior\footnote{Cf. [13.45, 13.46, 13.71], for introductory sources to the multifractal formalism, and [13.10], for more details on the structure function approach introduced below.}, a feature also encountered in data from turbulent flows. In fact, the first attempt at recovering traces of multifractality also dates back to the early 1990s [13.160]. A few years later, this topic has been taken up by several other groups (see for example [13.14, 13.59, 13.80, 13.136, 13.155, 13.156]).

Let us review and illustrate the typical physical approach for determining multifractal structure within a dataset following the lines of [13.80, 13.155, 13.156]. In order to establish deviations from monofractal behavior, one usually investigates the scaling of the moments of a signal $x(t)$, the time series of log prices in our case. One considers the cumulated sum $y(t)$ of the signal, which is also called the ”landscape” of the signal, and calculates the height-height correlation (or structure) function $c_q(\tau)$:

$$c_q(\tau) = E \left[ |y(t + \tau) - y(t)|^q \right].$$

(13.20)

For arbitrary $q$, a generalized Hurst exponent can be defined through the relation:

$$c_q(\tau) \sim \tau^{qH_q}.$$  

(13.21)

If a linear dependence is obtained, i.e. $H_q$ turns out to be constant, then the signal is monofractal or, in other words, the data follows a simple scaling law for all values of $q$. This would hold for fractional Brownian motion. However, with a nonlinear development of $H_q$, the data generating process is more complicated and will be said to be of a multifractal nature. The behavior of the underlying process can, then, only be characterized fully by the whole spectrum of its local Hurst exponents at various powers $q$. Since some curvature of the $H_q$ spectrum is a necessary condition, one may also extract information from the local slopes. For example, Vandewalle and Ausloos [13.155] report the estimate of $C_1 = -\frac{dH_q}{dq}\bigg|_{q=1}$ (known as the intermittency or sparseness parameter) for various exchange rates.

Examples of empirical scaling exponents of financial data (i.e. the development of the exponent $qH_q$ in (13.21)) are exhibited in Fig.13.3. This plot nicely confirms that some curvature in the scaling exponent can be found for various time series from different types of financial markets. The deviations from the linear development expected under uni-fractal Brownian motion suggest that
the data under consideration are characterized by multifractal behavior. Since this result is shared by all the studies mentioned above, the multifractal nature of financial returns may be added as a new stylized fact, which extends and generalizes prior insights on the temporal characteristics of both returns and volatility.

Disturbingly, none of the time series models that are currently used in the applied financial literature is capable of systematically reproducing this property of empirical data. However, whether spurious multifractal behavior could originate from simpler models seems to be an open question. While [13.155,13.156] report negative results from Monte Carlo simulations of ARCH models, Baviera et al. [13.12] found transient behavior of ARCH models resembling that of a nonlinear $H_q$ spectrum. Furthermore, Berthelsen et al. [13.16] have already demonstrated that short time series from random walks may appear to exhibit multifractality and [13.22] give another example of a monofractal process that shows apparent multiscaling. Much future research is, therefore, needed to single out which types of processes could conform to the seemingly universal finding of a nonlinear $H_q$ spectrum in financial returns.

If multiscaling does not turn out to be a spurious characteristic of 'simpler' time series models, one of the major contributions of physicists to finance may consist in their development of asset pricing models with genuine multifractal
behavior. First important steps in this direction seem to have been made with the application of combinatorial models of multifractal cascades to financial data (cf. \cite{13.50,13.113,13.114,13.136}). Interestingly, Mandelbrot et al. \cite{13.114} provide a model in which a multifractal cascade acts like a time transformation, a concept that also appeared in various econometric models of asset returns. First experiments indicate that these models (originating from the physical literature on turbulence) seem to be able to outperform GARCH type specifications in terms of the fit of the unconditional distribution, although, essentially, they are one-parameter families of stochastic processes (cf. \cite{13.106}). The upper right-hand panel of Fig. 13.0 provides an illustration of an artificial price series generated from a stochastic process with a multifractal time transformation. An alternative approach building upon physical models of turbulence is due to Moffat \cite{13.123}, who developed a dynamical (behavioral) theory of capital markets based on a continuous time description similar to that for hydrodynamic flow. He determined the onset of turbulent behavior and showed that intermittency exists for the time series spectra of volatility distributions in this model. A certain drawback of the multifractal apparatus has been the combinatorial nature of the models available from statistical physics literature which does not enable one to formulate forecasts based on an estimated model. However, recent developments of iteratively soluble stochastic cascade models seem to be able to overcome this problem allowing for short-term forecasts of volatility based on the multifractal properties of past data \cite{13.23,13.56}.

It appears noteworthy that many physicists seem to have been fascinated (and motivated to approach financial data) by the similarity between certain empirical characteristics in finance and turbulence. These similarities seem to have been noticed first by Vassilicos \cite{13.159}. Further contributions are \cite{13.59,13.73,13.76,13.118}. In fact, a visual inspection of time series plots of both financial returns and velocity differences in turbulent flows shows perplexing similarities (cf. \cite{13.118}). Similarities between both types of data include the phenomenon of volatility clustering and the leptokurtotic shape of the unconditional distribution. On the surface we find, in fact, two of the most elementary ‘stylized facts’ of empirical finance to be shared by turbulent dynamics! However, although far-reaching analogies have been drawn on the basis of these similarities \cite{13.59}, it soon turned out that there are also some important differences: (i) while price changes are almost uncorrelated (over time spans longer than a few minutes), velocity changes do exhibit significant antipersistence as is apparent from their typical diffusion exponent \(\sim 1/3\) \cite{13.118}, (ii) the approach to the Gaussian shape under time aggregation appears to occur much faster with turbulence data than with financial returns. The latter feature (which might be due to a higher degree of long-term dependence in volatility in financial data) led Holdom to conclude: ‘The
turbulence in the financial markets is stronger in this sense than the turbulence in the fluid’ [13.76]. Hence, although there are indeed interesting (and stimulating) similarities, the every day notion of ‘turbulence’ in financial markets does not extend to all quantitative details of the physical phenomenon. Whether the mathematical routes to multifractal or multi-affine behavior are similar in turbulence and finance, therefore, remains an open question.

This ends our discussion of empirical scaling laws in financial data and we now turn to attempts towards an explanation of these statistical characteristics.

13.5 Explanations of Financial Scaling Laws

13.5.1 Methodological Background

As already mentioned several times in the preceding sections, the main tenets of the efficient market hypothesis can also be interpreted as hypotheses on the origin of the statistical characteristics of prices and returns: if prices follow a martingale process and their increments reflect forthcoming information on future earning prospects in an immediate and unbiased manner, then the distribution of returns must be a mere reflection of the distribution of news. Hence, the news arrival process would have to share the features of fat tails and volatility clustering and should even exhibit multifractal features. It is worth emphasizing that under this hypothesis the origin of scaling laws in financial time series is to be found in exogenous forces that cover a wide variety of influences (e.g. climatic and political factors). Only part of the complex bundle of news factors results from economic variables in a strict sense (e.g. macroeconomic factors) and an explanation of the scaling laws would, therefore, lead us outside the realm of economics. Furthermore, this implication of the EMH cannot be tested as the news arrival process is not observable itself.

Although one should surely not deny the paramount importance of news for the price formation in financial markets, one may argue that a perfect one-to-one relationship between news and returns appears too farfetched. News are incorporated into prices through the interplay of demand and supply: in most cases new information will be available to some market participants prior to others. If the news is favorable, the better-informed traders will buy additional units of the asset in order to gain from their informational advantage. This will drive up prices and thereby incorporate the knowledge of the insiders into prices. Mutatis mutandis, the same should happen in the case of adverse information hitting the market. However, although we might suspect that the ‘invisible hand’ of the market mechanism may operate relatively efficiently and
smoothly most of the time, we also have evidence of various deviations from purely information-driven price formation. Much of the scattered empirical work on this topic has been mentioned in Sect. 13.3.1 above.

From a common sense aspect, one may argue that one usually observes a quite diverse number of behavioral variants among traders with informational trading constituting only part of the overall picture. With a large fraction of agents following chartist practices, large companies pursuing portfolio insurance and synthetic hedging programs, and some traders arguably being guided by herd instincts, we have a broad range of trading motives in real-life markets. Since large fractions of demand and supply originate from these diverse backgrounds, they will also exert an influence on the price formation process. The question, then, is whether all these factors (agents) only add some slight amount of noise to the efficient formation of prices, or whether, on the contrary, they are crucial to the resulting market outcome. The latter view would, of course, imply that markets are subject to some kind of endogenous dynamics originating from the interaction of individual traders. The perplexing similarity of the statistical characteristics of very different markets could, then, be explained by the similarity of the behavior of traders. The idea that the financial scaling laws have their origin in the trading process with its interaction of a large ensemble of heterogeneous traders has been denoted the interacting agent hypothesis (cf. [13,109]). Its implementation requires to construct structural behavioral models of financial markets whose stochastic characteristics match the stylized facts found in real-life data.

Unfortunately, standard modeling practices in economics have rather tried to avoid heterogeneity and interaction of agents as far as possible. Instead, one often restricted attention to the thorough theoretical analysis of the decisions of one (or a few) representative agents. Since, in recent years, the consciousness of the importance of heterogeneity has been increasing, a couple of papers have appeared that implement various types of microscopic behavioral models of financial markets.

For physicists, the explanation of scaling laws through the collective behavior of a multitude of elementary units (traders in our case) is a rather familiar approach. It is, therefore, not astonishing that quite a number of multi-agent models have been developed by physicists, partly in collaboration with economists and partly without contact to the economics community. Interest in modeling markets arose from the recognition of the universal character of scaling laws in finance and their resemblance to scaling laws found in various physical multi-unit systems. The resulting research program is described in [13,141] as follows:
“Statistical physicists have determined that physical systems which consist of a large number of interacting particles obey universal laws that are independent of the microscopic details. This progress was mainly due to the development of scaling theory. Since economic systems also consist of a large number of interacting units, it is plausible that scaling theory can be applied to economics.”

Quite at the same time, a number of authors in economics proposed a statistical approach for modeling micro-economic heterogeneity among agents [13.2, 13.3, 13.85, 13.101, 13.132]. As stressed by the last author, such an approach bears implications very different from the formerly dominant representative agent methodology: while analysis and results are the same at both the micro and macro level with representative agents, the statistical approach may lead to emergent properties on the macro level that result from the interaction of microscopic units and cannot be inferred from the observable characteristics on the micro level themselves. Interestingly, one of the arguments in [13.104, 13.132] is that a multi-agent approach may allow an endogenous explanation of time-varying variances.

13.5.2 Variants of Microscopic Models

In economics, the first microscopic models\textsuperscript{13} have not been developed so much to explain the stylized facts as emerging properties of an artificial market. Rather, their focus has been on learning of agents from experience and whether learning leads agents to converge with their plans to some theoretical equilibrium of the model. Typical examples are the papers [13.4, 13.29] and the Santa Fe stock market model [13.90, 13.127]. All of these examples use either genetic algorithms (GAs) or genetic programming (GP) to formalize the information processing of individual traders. Arifovic [13.4] builds upon a well-known static model of a foreign exchange market. She encodes agents’ choice variables (consumption and holding of foreign currency) by binary numbers and applies the usual GA operations of selection, crossover, and mutation to these simple ‘strategies’. Over time, successful behavior is on average maintained, whereas unsuccessful agents will adapt the ‘strategies’ of the more lucky ones. Mutation also allows to try new variants which are not in the pool of alternatives

\textsuperscript{13} Even prior to the studies mentioned in the main text, Stigler [13.144] applied elementary Monte Carlo models in a simulation of a financial market. The early microstructure literature covers some more examples of microsimulations (e.g. [13.31, 13.69]). Kim and Markowitz [13.84] simulate markets with an ensemble of investors pursuing different versions of portfolio insurance strategies. They investigate whether this type of activity leads to excessive volatility, a question which has been raised after the 1987 stock market crash.
at time zero. One of the main conclusions of this model is that the time series from the simulations share the random walk character of empirical data and also allow to explain some features that emerged in experimental markets (laboratory markets with a small number of human participants interacting via computer).

Chen and Yeh's work [13.28,13.29], on the other hand, has agents forming a theory on the relevant variables via GPs. That is, a given input time series (for example, the price record) is fitted by varying functional forms which are developed out of a tree-like structure from some elementary operations (+, -, /, x, cos, sin, etc). The emerging population of genetic programs is subject to similar processes of selection, crossover, and mutation like the simpler genetic algorithm agents of Arifovic [13.4]. In early papers, e.g. [13.28], the main focus is on the convergence of the market to its theoretical equilibrium. In the most recent extension of their model, Chen and Yeh [13.29] distinguish between two subsystems: traders who use GPs in order to predict prices and volatility and business schools who produce and supply these GPs in some kind of academic competition. Among other things, they show that often the relative price changes from the simulations appear to be uncorrelated. Hence, the martingale property of prices seems to emerge without traders actually believing in such behavior.

Finally, we mention the Santa Fe stock market model which has been elaborated by a number of authors (e.g. [13.82,13.90,13.127]) over more than a decade. The structure of this model is even more complex in that GAs are combined with classifier systems. The latter part contains diverse trading signals which include both fundamental factors (e.g. relationship between price and dividends) and chartist factors (e.g. moving averages of various length). From the activated classifiers (whose if-part conforms with the environment), one is chosen according to a random draw with past success governing the probabilities of activation. The prescribed trading leads to a change of individual wealth and also allows to update the strength of the classifier. Again, classifiers are subject to the usual GA operations after a certain number of trading periods. In the simulation experiments, both the macroscopic development of prices and volume in this artificial market as well as the rate of activation of various classifiers are observed. The first important finding was a bifurcation in the model dynamics when changing the frequency of reevaluation of the classifiers (via the GA mechanism); with more frequent revaluation, a dominance of chartist devices emerges while with less frequent application of GAs, the traders learn to prefer fundamentalist techniques. In a recent paper [13.90] the capability of the Santa Fe model to match some of the stylized facts of financial markets (volatility clustering, leptokurtosis) is emphasized. As only some rough statistics are provided, it is, however, not entirely clear so far how closely this popular model is able to mimic and explain the stylized facts.
Much simpler interaction models (from the economist’s point of view) have been proposed by physicists: [13.32, 13.79, 13.142, 13.143] all construct models based on lattice structures well known in percolation theory and Ising type models. Denoting the states of each agent (knot) as ‘buying’, ‘selling’ and ‘inactive’, dynamics arise from cluster formation and activation processes that are propagated through the system. Since such models are well-known to produce power-laws at least in the vicinity of a critical point (the percolation threshold), it is not astonishing that modeling market interaction in this way leads to similar results. Recent papers have developed this approach further by introducing features to produce volatility clustering as well as power-laws close to the numerical values obtained with empirical data. As this branch of literature is authoritatively surveyed in the companion paper by Somerette, Stauffer, and Takayasu (see Chap.14), we dispense with a detailed treatment here. Another approach with a rather direct inspiration by physical models is to be found in a pioneering paper by Takayasu et al. [13.147]. Here, the behavior of traders is formalized using threshold values for the decisions to buy or sell the asset. Agents who have just carried out a transaction, adjust their threshold values before participating in further trading rounds. In Takayasu et al. [13.147] thresholds are increased (decreased) if the agent had bought (sold) in his last transaction. Simulated time series show that these threshold dynamics can generate some realistic features like crashes in prices. A somewhat similar model is developed in Bak et al. [13.9], where a new threshold is chosen randomly from those of the remaining agents. Bak et al. show theoretically that their model leads to a self-similarity parameter $H = 0.25$ for prices, which, of course, is in contrast to empirical results. In an extended version time dependence of price changes is built into the model through some kind of volatility feedback (thresholds are now also updated by an amount proportional to past price movements). Not entirely surprising, this leads to measurable long-term dependence in price changes with an estimated $H = 0.65$. Although this appears to conform to some results of the older empirical literature, it should be kept in mind that these findings have been disputed in more recent research and, thus, the long-term dependence in price levels resulting from the volatility feedback, seems also to be at odds with empirical findings (cf. Sect. 13.3.3).

In personal communication, we found very different reactions from scientists from the economics and physics communities to these models. While, on the one hand, many physicists would consider the first type of models unnecessarily complicated, economists, on the other hand, would bemoan the almost entire absence of key economic concepts like utility or profit maximization, as well as the neglect of the issue of expectation formation, in the physicists’ models. Nevertheless, these approaches are often considered as very useful and necessary steps towards incorporating new modeling techniques and new per-
spectives on interaction in markets. However, in order to take into account the peculiarities of economic systems (namely, in order to account for the non-mechanical nature of individual decisions even within a multi-agent framework), it seems indispensable to attempt a synthesis of both approaches.

The models by Levy, Levy, and Solomon [13.92–13.94] and Lux and Marchesi (variants of which have been published in from both economics and physics journals) may constitute some modest steps in this direction. [13.92–13.94] allow agents to split their wealth between a safe bond with constant interest rate and a risky asset with stochastic returns and derive their decisions from a standard utility-of-wealth function. Prices are determined from a market clearing condition (so that demand equals supply in any period). With explicit wealth maximization, expectation formation of agents about future prices has to be included in the model. This is formalized in a relatively simple manner by assuming that agents have different memory spans. Each agent, then, computes the expectation of next period’s price as the average of past prices observed over the length of his time horizon. For many choices of parameters, the model produces very spectacular (but not entirely realistic) crashes and booms of the market price [13.92]. Furthermore, varying combinations of time horizons among traders lead to diverse and sometimes surprising results for the time development of the share of wealth owned by different groups [13.93]. The existing papers are silent on the scaling laws for prices and returns, but Levy and Solomon [13.94] show that with the (crucial) addition of ‘social security’ paid out to investors whose wealth falls below some threshold value in the course of events, realistic power-laws (in accordance with Pareto’s famous law) for the distribution of wealth among agents can be obtained.

The simulation framework of Lux and Marchesi [13.109,13.110] builds upon earlier theoretical work [13.101,13.104,13.105] in which the features of simple multi-agent models are analyzed via mean field approximation and master equation techniques. The model contains two groups of traders: fundamentalists who buy (sell) when the price is believed to be below (above) the fundamental value, and chartists, who follow trends and are subject to some sort of herd behavior. The latter group is divided into two subgroups: optimistic individuals who believe in an increase of the price in the near future and pessimistic ones who believe that the price will fall. Furthermore, agents are not bound to remain within one group for all time, but are allowed to change behavior by switching between these three behavioral variants. Switches between the optimistic and pessimistic chartist group are governed by the joint influence of the observed (temporary) price trend and the observed majority opinion among their fellow traders: rising prices together with an optimistic majority will induce former pessimists to join this majority and vice versa. On the other hand, with conflicting signals (for example, rising prices together with
a pessimistic majority), the incentive to change behavior will be considerably weaker. Switches between the chartist and fundamentalist groups are governed by the difference of profits earned by traders from both groups. Of course, if one strategy has an advantage, traders from the other group will be attracted to this alternative. Given the distribution of traders over the three groups at any instant in time, their demand and supply decisions follow from their particular strategy and their opinion about the future development of the market. If imbalances between demand and supply result, prices are adjusted in the usual manner. With changes in market prices, trends, majority opinions, and profitability of strategies may also undergo changes, which leads to revisions of agents' strategies and a new distribution over groups.

As Lux and Marchesi [13.109] are mainly interested in the emergence of scaling laws from the market interaction, they perform the following 'experiment': changes of the fundamental value of the asset are introduced as external input. However, this 'news arrival process' is assumed to lack all the typical characteristics of real-life data (fat tails and heteroscedasticity), but is instead modeled as a white noise process. According to the efficient market hypothesis, returns should reflect this 'innocuous' distribution of the news, whereas endogenous emergence of scaling laws would lend some support to the interacting agent hypothesis. In fact, it turns out that scaling laws with realistic exponents emerge for a wide range of model parameters. Hence, as illustrated in a typical simulation in Fig.13.4, it appears that the market interactions of agents transform exogenous noise (news) into fat tailed returns with clustered volatility. The upper left-hand panel of Fig.13.0 shows a typical price path from a simulation of the model which also is hard to distinguish from empirical records. The paper also provides an avenue towards reconciliation of the seemingly adverse views of the market efficiency and the interacting agent approaches in that prices are shown to closely follow the motion of the fundamental factors. In fact, a number of standard tests for the implications of the efficient market hypothesis (e.g. unit root tests) proved unable to recover any 'inefficiency' in the price series from this artificial market. Hence, the emergence of scaling laws does not come along with deviations from efficient price formation that are easily recognized.

13.6 Conclusion

In this chapter, we have surveyed research on scaling phenomena in financial data pursued by physicists and have compared their methodology and results with the approach of economists dealing with the same topic. We have also tried to put this work into perspective by discussing how far it is reconcilable

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14 This even happens when we assume a stationary fundamental value, cf. [13.110].
Fig. 13.4. A typical time series of returns from the model of Lux/Marchesi [13.100] (a) compared with the empirical record of daily returns of the German share price index DAX (b). Both time series share the clustered volatility phenomenon and a higher concentration of large returns than would occur under a normal distribution. As shown in [13.100] for the artificial financial market, both the tail index estimates and the estimates of the statistics for long-term dependence are in good agreement with empirical results.

with traditional models in finance (the efficient market hypothesis) or whether it leads to a new viewpoint on market interactions.

As it turns out, many of the scaling phenomena highlighted in this literature (i.e. the cubic power-law of the empirical distribution of returns or the power-law in temporal dependence of squared and absolute returns) have been in no way alien to economics, but under the headings of ‘fat tails’ and ‘volatility clustering’ count as ubiquitous stylized facts of financial data. In most cases, there are no material differences in the empirical results reported by researchers from economics and finance-afficionados from physics (the convergence of initially different outcomes has been pointed out above). The key difference is rather that both groups often tend to express one and the same phenomenon in a somewhat different language. Right here the danger emerges, that parallel research remains unnoticed and earlier investigations by researchers from the other tradition are repeated. The history of the Lévy distribution hypothesis and its abandonment provides an example of this possibility.
However, more recently, the focus of analysis has been broadened and new work on additional scaling laws has been carried out. Interesting examples include the analysis of scaling in the distribution of intra-daily transaction numbers [13.130] or scaling in waiting times in tick-by-tick data [13.131] which unambiguously add new insights to our body of knowledge on empirical characteristics of financial data. Given the currently burgeoning interest in high-frequency data in finance [13.62], the increased availability of large datasets at intra-daily frequencies, and the experience of physicists to deal with such large datasets, in fact, suggest that a lot remains to be detected by such data analytical exercises.

From the conceptual point of view, the major innovative contributions by physicists are, however, not so much in the purely empirical work, but in suggesting new approaches in the modeling of the underlying phenomena: first, the introduction of the multifractal framework allows a unifying treatment of the different scaling laws in various powers of returns\textsuperscript{15} and may provide us with a clue to a new vintage of empirical asset pricing models able to account for these phenomena. Second, from the theoretical side, the main contribution lies in the recognition of statistical physics, that scaling laws like the ones found in financial markets can often be explained by interactions in multiparticle systems. For a long time, such a perspective has indeed been alien to economics with its paradigm of the representative agent, but it seems to be gaining ground. The recent attempts at designing multi-agent models of financial markets that were sketched in the last part of our survey may pave the way to a new branch of descriptive models for financial markets.

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\textsuperscript{15} With respect to these features Clive Granger wrote: “I feel that when a satisfactory theory for this area is found, it may unlock a rush of new results having real practical importance.” [13.65]
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