Strategic asset allocation with heterogeneous beliefs

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In this paper, I show how the presence of agents with heterogeneous beliefs generates the price trends observed in the financial markets. I develop an asset pricing model in which agents have long horizon objectives, based on a stream of consumption. Each agent chooses a forecasting model and maximises a recursive utility function. The choice of the forecasting model in each period determines the agent type. However their types change over time according to the relative performance of the forecasting models. This happens because agents have an incentive to adopt the forecasting model with the best performance in the previous period to coordinate with the market. I estimate the asset pricing model using data on the international stock markets. The exercise shows that especially for very risk averse individuals, the accounting for the intertemporal hedging demand is crucial.

This paper bridges the literatures on intertemporal asset allocation and on heterogeneous beliefs. I assume that agents with long horizon objectives, based on a stream of consumption, maximise a recursive utility function. In order to do that, they choose a return forecasting model. They make this choice in each period and this determines their types. These types change over time based on the relative performance of the forecasting models. This happens because agents have an incentive to adopt the forecasting model with the best performance in the previous period to coordinate with the market. However, they do not receive perfect information regarding the performance of the strategies.

From the intertemporal asset allocation framework, the asset pricing model inherits the ability to reproduce the behaviour of consumption-based utility maximizing investors with long horizon objective functions. I solve the intertemporal asset allocation problem introduced by Merton (1969) and Samuelson (1969) using the approximate solution of Campbell, Chan and Viceira (2003). I use the class of preferences in Epstein and Zin (1989, 1991) and Weil (1989) in order to individuate the agent’s risk aversion and elasticity of intertemporal substitution. This framework is convenient because it allows to solve the portfolio selection problem in the presence of multiple risky assets. This is in spark contrast with the usual myopic mean-variance framework with a single risky asset in the literature on heterogeneous beliefs (e.g., Brock and Hommes (1997, 1998) or Boswijk,

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By assuming heterogeneous beliefs I am able to better describe the individual and market behaviours and, as such, reproduce the stylized facts of asset returns. There are many attempts in the literature to reproduce these effects. Nevertheless, none of them is able to fully solve all the puzzles and explain the momentum and value effects at the same time. The biggest challenges in most cases are solving the equity premium puzzle and generating the momentum effect. Models with heterogeneous beliefs are able to conciliate momentum and value effects as well as to generate higher volatility in returns. These are achieved because the agents change beliefs (i.e., forecasting models) over time.

Heterogeneous beliefs models alleviate the equity premium puzzle. Abel (1989) notes that heterogeneity per se does not necessarily invalidate the representative agent approach, but heterogeneity in beliefs does. This happens because the cross-sectional distribution of expectations cannot be summarized by a single sufficient statistics. Abel (1989) also shows that introducing heterogeneity in beliefs can substantially increase the equity premium (see also Basak (2005)).

The formulation matches several theoretical and empirical evidence of heterogeneity of expectations. It also matches the evidence of variability over time in the choice of forecasting models as in Frankel and Froot (1987) for instance. In addition, the assumption of heterogeneous agents avoids a no-trade equilibrium that arises as a consequence of theorems such as those in Milgrom and Stokey (1982).

Kandel and Pearson (1995) and Bamber, Barron and Stober (1999) provide evidence of heterogeneity in analyst expectations for stocks regarding earnings around announcements. Analysing bubbles, Shiller (2002) provides evidence of heterogeneity in the expectations of the future performance of the market. Finally, Patton and Timmermann (2010) study the sources of disagreement about forecasts of macroeconomic variables and find that they are persistent and indicate that they stem from heterogeneity in priors or models, not different information sets.

Frankel and Froot (1987) and Taylor and Allen (1992) report survey evidence of heterogeneity in expectations. In particular, Frankel and Froot (1987) find that forecasting companies use different models to project returns, and that the number of companies using different classes of models changes over time. Surveying exchange rate expectations of financial specialists, Menkhoff (1997) shows that investors tend to use different trading strategies. Their strategy choice depends on the investment horizon they are trying to forecast. They basically use chartist strategies in the short run, and keep fundamentalist strategies for long horizons.

In this paper, I use the approximate solution of Campbell, Chan and Viceira (2003) to calculate the demands for assets of each agent type, and apply the framework of Brock and Hommes (1997, 1998) and Boswijk, Hommes and Manzan (2007) to model the evolution of types. Modelling the evolution of types corresponds to describing how the proportions of agents using a given return
forecasting model evolve over time. Therefore, I extend the models of Brock and Hommes (1997, 1998) and Boswijk, Hommes and Manzan (2007) to consider also long term investors.

The agents adjust their forecasts trying to match what they believe to be the dominant forecasting strategy in the market. I assume that the market is populated by many agents choosing among different forecasting models. These agents are aware that other agents are also choosing their models in the same way. Therefore, they know that the most accurate forecast is the one given by the strategy chosen by the majority of them. This happens regardless of the theoretical support that a given model may enjoy. In addition, all agents doubt about the rationality of the other agents.

In the empirical section, I assume the perspective of an investor in the U.S.A. who would like to diversify his/her portfolio using the international stock markets. For simplicity, I assume that there are only two agent types in each market: fundamentalists and chartists. Fundamentalists use value strategies and chartists use momentum strategies. I estimate the model using stock market data from the U.S., the U.K., Japan, and Hong Kong. I start by estimating a simple dividend-price factor model and a simple momentum factor model for each of these four markets. Next, I use these factor models as the fundamentalist and chartist strategies and examine the resulting dynamics.

Given these forecasting models, I obtain the demand for assets of each agent type. I use these demands to compute the relative performances of their strategies. These performances determine the fraction of agents using each strategy. I show that myopic and long-term investors have different demands for assets and, therefore, different performances. I also show that the investment horizon has different effects on the demand for assets of fundamentalists and chartists. The component of the demand for assets that is ignored in a myopic framework can be significantly large and impact the estimation of the proportions of agents. This is especially true when agents are very risk averse. In fact, in this case the omitted term in the myopic framework can be the dominant one in certain markets. The agent’s decision of using a fundamentalist or chartist forecast in these markets will often depend on whether he/she believes that agents are myopic or not. In addition, I show that the level of noise in the observed performances also has different impacts on the model results whether we consider the complete intertemporal demand for assets or only its myopic component.

The paper is organised in three sections following this introduction. In Section 1, I derive the asset-pricing model and discuss its theoretical results. In Section 2, I estimate the model and analyse the results focusing on the differences between the intertemporal solution and the myopic solution previously obtained in the literature. Section 3 concludes.
I. The model

There is an infinite number of long-term investors of \( H \) different types. The trading strategy used to forecast returns determines the agent type \( h \). In most of the paper, I restrict the analysis to \( H = 2 \) (i.e. fundamentalist or chartist types), but I develop the model for the general case with a given number of types \( H \). Agents extract information from prices: they switch between trading strategies (change their types) as they respond to the previous performance of the strategies. However, they do not receive perfect information regarding the performance of the strategies. They all have access to the same information set but use different return forecasting models. Therefore, I model differences in opinions (i.e., forecasting models), and not differences in information sets.

A. The investor’s maximization problem

Time is discrete, and investors that live infinitely maximise the recursive preferences defined over a stream of consumption, as described by Epstein and Zin (1989, 1991) and Weil (1989).\(^1\) There are \( n \) risky assets in the economy, and investors allocate their wealth among these assets and consumption. The investor’s problem is to choose the portfolio allocation, \( \alpha_{h,t} \), and consumption, \( C_{h,t} \), that maximises his/her utility at every time \( t \) given his/her type. Each investor is, however, restricted by a budget constraint.\(^2\) So, their problem is given by:

\[
\begin{align*}
\text{arg} \quad \max_{\alpha_{h,t} \in \mathbb{R}^n, C_{h,t} \in \mathbb{R}} & \quad U(C_t, E_t[U_{t+1}]) = \left[ (1 - \delta)C_t^{1-\gamma} + \delta(E_t(U_t^{1-\gamma}))^\frac{1}{1-\delta} \right]^\frac{\theta}{1-\theta} \\
\text{s.t.} & \quad W_{t+1} = (W_t - C_t)(1 + R_{p,t+1}), \\
& \quad R_{p,t+1} = \sum_{i=2}^{n} \alpha_{h,i,t}(R_{i,t+1} - R_{1,t+1}) + R_{1,t+1}.
\end{align*}
\]

where \( C_t \) is the agent’s consumption and \( E_t(.) \) is the agent’s conditional expectation operator at time \( t \). The agent’s relative risk aversion coefficient is \( \gamma > 0 \), \( \psi > 0 \) is the agent’s elasticity of intertemporal substitution coefficient, \( 0 < \delta < 1 \) is the agent’s time discount factor and \( \theta \equiv (1 - \gamma)/(1 - \psi^{-1}) \). In the consumption-based budget constraint, \( W_t \) is wealth at time \( t \), and \( R_{p,t+1} \) is the portfolio return on the next period. Finally, \( \alpha_{h,i,t} \) is the portfolio weight on asset \( i \) at time \( t \) and \( R_{i,t+1} \) is the return on the next period. The first asset \( (i = 1) \) is proxy for a risk free asset with a real return of \( R_{1,t+1} \).

\(^1\)The power utility is a special case of the Epstein-Zin function. We can obtain it by letting \( \gamma = \psi^{-1} \) (and hence \( \theta = 1 \)). In addition, the log utility is a special case of the power utility, it can be easily obtained by adding the restriction \( \gamma = 1 = \psi^{-1} \). With time varying investment opportunities, this condition generates the myopic portfolio allocation. However, as Giovannini and Weil (1989) showed, \( \gamma = 1 \) or \( \psi^{-1} = 1 \) alone are not sufficient for this result.

\(^2\)Because the maximisation problem is the same for every agent type, I do not write the subscripts here to simplify the notation.
Epstein and Zin (1989, 1991) find that solving the problem in (1) results in the Euler equation:

\[
E_t \left\{ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\theta}} \right\} (1 + R_{p,t+1})^{-(1-\theta)}(1 + R_{i,t+1}) = 1
\]

that must hold for any asset \( i \), (including the portfolio \( p \)) along the optimum consumption path. The equation shows the relationship between portfolio allocation, consumption and expectations (or \textit{beliefs}). It highlights the importance of the forecasting model used by the agent. The forecasting model impacts both on the planned growth in consumption and also on the asset allocation.

In general there is no closed form solution to this problem. I, thus, apply the same approximate solution of Campbell, Chan and Viceira (2003) to obtain an expression for the asset’s demand of each agent type. I describe the procedure in details in the following subsections. It begins by postulating that agents describe the dynamics of the relevant state variables as a first-order vector auto-regressive process \textit{VAR}(1). Campbell and Viceira (1999) shows that the approximate solution exists if the elasticity of intertemporal substitution is close to 1. We then log-linearize the portfolio return and budget constraints in (1) as well as the Euler equation in (2) close to this value. This step produces an expression for the expected excess return of each asset. Next, we write everything in terms of the state variables in the \textit{VAR}. Solving for the consumption and portfolio rules, we finally obtain the optimal asset demand for each investor type. See Appendix A for a basic derivation of excess returns using a stochastic discount factor framework.

**Dynamics of returns.** — Formally, I define

\[
x_{t+1} = \begin{bmatrix} r_{2,t+1} - r_{1,t+1} \\ r_{3,t+1} - r_{1,t+1} \\ \vdots \\ r_{n,t+1} - r_{1,t+1} \end{bmatrix},
\]

where \( r_{i,t+1} = \ln(1 + R_{i,t+1}) \) \( \forall \ i \), and \( x_{t+1} \) is a vector of excess returns. I also include other state variables \( s_{t+1} \), such as the price-earnings ratio, realised returns or other return forecasters, stacking \( r_{1,t+1}, x_{t+1} \) and \( s_{t+1} \) into an \( m \times 1 \) vector \( z_{t+1} \):

\[
z_{t+1} = \begin{bmatrix} r_{1,t+1} \\ x_{t+1} \\ s_{t+1} \end{bmatrix}.
\]
A fundamentalist agent will model the market dynamics by considering fundamentalist predictors. Chartists will decide based exclusively on past returns. The difference between them is in the coefficients of the VAR:

\[ z_{h,t+1} = \phi_{h,0} + \phi_{h,1} z_t + v_{h,t+1}. \]

The trading strategy that agent \( h \) is actually using determines the coefficients \( \phi_{h,0} \), the \( m \times 1 \) vector of intercepts, and \( \phi_{h,1} \), the \( m \times m \) matrix of slope, with shocks \( v_{h,t+1} \) that satisfy

\[ v_{h,t+1} \sim i.i.d. \ N(0, \Sigma_{h,v}), \]

\[ \Sigma_{h,v} \equiv Var_t(v_{h,t+1}) = \begin{bmatrix} \sigma_{h,1}^2 & \sigma_{h,1x}' & \sigma_{h,1s}' \\ \sigma_{h,1x} & \Sigma_{h,xx} & \Sigma_{h,xs}' \\ \sigma_{h,1s} & \Sigma_{h,xs} & \Sigma_{h,ss} \end{bmatrix}. \]

These distributional assumptions allow for a cross-sectional correlation between the shocks, which are otherwise iid over time. Given the homoskedastic VAR(1) formulation, it is easy to derive the unconditional distribution of \( z_{t+1} \) because it inherits the normality of the shocks. Note that unlike Brock and Hommes (1998), I assume that agents may also disagree on how to estimate these variances and covariances.

**Approximate solution.** — Epstein and Zin (1989, 1991) show that it is possible to write the value function obtained from the maximization in (1) per unit of wealth as a power function of the optimal consumption-wealth ratio:

\[ V_t \equiv \frac{U_t}{W_t} = (1 - \delta)^{-\frac{\psi}{1-\psi}} \left( \frac{C_t}{W_t} \right)^{\frac{1}{1-\psi}}. \]

Campbell and Viceira (1999) note that, under the assumptions made here,

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3 The homoskedasticity assumption is rather restrictive because it rules out the possibility that state variables predict changes in risk. This means that they can only affect the portfolio choice by predicting changes in expected returns. However, many previous studies show that the effect of those risk changes over portfolio choice is limited. Campbell (1987), Harvey (1991) and Glosten, Jagannathan and Runkle (1993) found only modest effects that are dominated by the effects of the state variables on expected returns. Also, Chacko and Viceira (2005) show that changes in risk are not persistent enough to have large effects on the intertemporal hedging demand.
which guarantees that the value function (8) has a finite limit as \( \psi \) tends to 1. This result is important because it allows for an approximation close to this limit where an analytical solution to the model exists.

Following Campbell and Viceira (2001) and Campbell, Chan and Viceira (2003), it is possible to approximate the return on the portfolio in equation (1). The approximation is exact in continuous time and very close to the true value at short time intervals. It is given by:

\[
\begin{align*}
(10) \quad r_{p,t+1} &= r_{1,t+1} + \alpha'_t x_{t+1} + \frac{1}{2} \alpha'_t (\sigma^2_x - \Sigma_{xx} \alpha_t),
\end{align*}
\]

where lower cases indicate variables in log and \( \sigma^2_x \equiv diag(\Sigma_{xx}) \) is a vector with the diagonal elements of \( \Sigma_{xx} \), i.e., the variances of the excess returns.

Similar to Campbell (1993, 1996), we can also log-linearise the budget constraint in the same problem. We do this around the unconditional mean of the log consumption-wealth ratio. This results in

\[
\begin{align*}
(11) \quad \Delta w_{t+1} &\approx r_{p,t+1} + \left(1 - \frac{1}{\rho}\right) (c_t - w_t) + k,
\end{align*}
\]

where \( \Delta \) is the difference operator; \( \rho \equiv 1 - \exp(E[c_t - w_t]) \); and \( k = \ln(\rho) + (1 - \rho) \ln(1 - \rho)/\rho \) is endogenous because it depends on the optimal level of \( c_t \) relative to \( w_t \). When \( \psi = 1 \), \( c_t - w_t \) is constant and \( \rho = \delta \). In this case, the budget constraint approximation is exact.

Applying a second-order Taylor expansion to the Euler equation in (2) around the conditional means of \( \Delta c_{t+1}, r_{p,t+1}, r_{i,t+1} \) gives way to

\[
\begin{align*}
(12) \quad 0 &= \theta \ln \delta - \frac{\theta}{\psi} E_t \Delta c_{t+1} - (1 - \theta) E_t r_{p,t+1} + E_t r_{i,t+1} \\
&\quad + \frac{1}{2} \Var_t \left[ -\frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{p,t+1} + r_{i,t+1} \right].
\end{align*}
\]

This log-linearised Euler equation is exact if consumption and asset returns are jointly lognormally distributed. This is the case when the elasticity of intertemporal substitution equals one (\( \psi = 1 \)).
noting that $\Delta c_{t+1} = \Delta(c_{t+1} - w_{t+1}) + \Delta w_{t+1}$ yields

$$E_t(r_{i,t+1} - r_{1,t+1}) + \frac{1}{2} Var_t(r_{i,t+1} - r_{1,t+1}) = \frac{\theta}{\psi} \gamma(\sigma_{i,c-w,t} - \sigma_{1,c-w,t})$$

$$+ \gamma(\sigma_{i,p,t} - \sigma_{1,p,t})$$

$$- (\sigma_{i,1,t} - \sigma_{1,1,t}),$$

where

$$\sigma_{i,c-w,t} = Cov_t(r_{i,t+1}, c_{t+1} - w_{t+1}),$$

$$\sigma_{1,c-w,t} = Cov_t(r_{1,t+1}, c_{t+1} - w_{t+1}),$$

$$\sigma_{i,p,t} = Cov_t(r_{i,t+1}, r_{p,t+1}),$$

$$\sigma_{1,p,t} = Cov_t(r_{1,t+1}, r_{p,t+1}),$$

$$\sigma_{i,1,t} = Cov_t(r_{i,t+1}, r_{1,t+1}),$$

$$\sigma_{1,1,t} = Var_t(r_{1,t+1}).$$

On the left hand side of (13), we have the average excess return of asset $i$ over asset 1 that each agent requires. We add one-half of the variance of the excess return because we consider log returns.\(^{4}\)

The factors that determine the required excess return on each asset are shown on the right-hand side. Factors that contribute to raise the risk premium are the excess covariance with consumption growth and excess covariance with the portfolio return. The last term cancels out when the asset is risk free. It relates the covariance of the asset’s excess return with the benchmark return to the required risk premium. Because consumption growth and portfolio return are endogenous, this is only a first-order condition describing the optimal solution. Thus, to solve the model, it is necessary to determine both those values.

Assuming that the optimal portfolio rule is linear in the VAR state vector but with a quadratic optimal consumption rule produces (14) and (15):

$$\alpha_t = A_0 + A_1 z_t,$$

$$c_t - w_t = b_0 + B_1^t z_t + z_t^t B_2 z_t.$$

\(^{4}\)The left-hand side of equation (13) is determined by the dynamics of $z_t$, which also determines the variances and covariances on the right-hand side. However, the second term ($\gamma(\sigma_{i,p,t} - \sigma_{1,p,t})$) is a function of portfolio choice, $\alpha_t$. This is calculated to make both sides equal for a given consumption policy. See Appendix A.
Here $A_0$, $A_1$, $b_0$, $B_1$, and $B_2$ are constant coefficient matrices with dimensions $(n - 1) \times 1$, $(n - 1) \times m$, $1 \times 1$, $m \times 1$, and $m \times m$, respectively, that we need to determine.

Now, we simply write the conditional moments that appeared in (13) as functions of the VAR and the unknown parameters in (14) and (15). Finally we solve for the parameters that satisfy (13).

For agent type $h$, we write the conditional expectation on the left-hand side of (13) as

$$E_{h,t}(x_{t+1}) + \frac{1}{2} Var_{h,t}(x_{t+1}) = H_x \phi_{h,0} + H_x \phi_{h,1} z_t + \frac{1}{2} \sigma_{h,x}^2,$$

where $H_x$ is matrix that selects the vector of excess returns from the full state vector, and $Var_{h,t}$ is the conditional volatility estimated by agent $h$ at time $t$.

Campbell and Viceira (2001) and Campbell, Chan and Viceira (2003) also show that it is possible to write the right-hand side of (13) as linear functions of the state variables:

$$\sigma_{h,c-w,t} - \sigma_{h,1,c-w,t} \equiv [\sigma_{h,i,c-w,t} - \sigma_{h,1,c-w,t}]_{i=2,3,...,n} = \Lambda_{h,0} + \Lambda_{h,1} z_t,$$

$$\sigma_{h,p,t} - \sigma_{h,1,p,t} \equiv [\sigma_{h,i,p,t} - \sigma_{h,1,p,t}]_{i=2,3,...,n} = \Sigma_{h,x} \alpha_{h,t} + \sigma_{h,1x},$$

$$\sigma_{h,1,t} - \sigma_{h,1,1,t} \equiv [\sigma_{h,i,1,t} - \sigma_{h,1,1,t}]_{i=2,3,...,n} = \sigma_{h,1x},$$

where $\nu$ is a vector of ones.

**The approximate demand for assets from agent $h$.** — By plugging (16) to (19) into the Euler equation (13) and solving for the portfolio rule, we finally obtain the optimal asset demand for each investor type $h$:

$$\alpha_{h,t}^* = \frac{1}{\gamma} \Sigma_{h,x}^{-1} \left[ E_{h,t}(x_{t+1}) + \frac{1}{2} Var_{h,t}(x_{t+1}) + (1 - \gamma) \sigma_{h,1x} \right]$$

$$+ \frac{1}{\gamma} \Sigma_{h,x}^{-1} \left[ -\frac{\theta}{\psi} (\sigma_{h,c-w,t} - \sigma_{h,1,c-w,t}) \right].$$
Equation (20) is the generalised multiple-asset demand of Restoy (1992) and Campbell and Viceira (1999) for agent type \( h \). It characterises the optimal portfolio choice as the sum of two components. The first one is exactly the myopic demand with many risky assets and lognormal returns. It does not depend on the elasticity of intertemporal substitution because this is a myopic component. The second is the intertemporal hedging demand term. With time-varying investment opportunities, the prediction in Merton (1969, 1971) is that an investor more risk-averse than a logarithmic investor would want to hedge against those shocks.\(^5\) We verify this by noting that the second term indeed depends on the excess covariance between the shocks on the return on the risky asset and the shocks on consumption growth. The investor demands more assets with returns that are negatively correlated with the consumption growth because he is willing to smooth consumption. This makes the intertemporal hedging demand term usually positive for such assets.

Equation (20) highlights the difference between the myopic and the intertemporal frameworks. In this equation, we see that the myopic term is only a fraction of the complete demand for assets. The intertemporal hedging demand term is the part that is ignored when we cast investment problems within a myopic framework.

Define the Intertemporal Hedging Demand at time \( t \) for agent type \( h \) as

\[
IHD_{h,t} = \frac{1}{\gamma} \Sigma_{h,xx}^{-1} \left[ -\frac{\theta}{\psi} (\sigma_{h,c-w,t} - \sigma_{h,1,c-w,t}) \right].
\]

Note that the hedging demand depends on \( h \) and can also vary over time. We can now rewrite (20) as

\[
\alpha^*_h t = \frac{1}{\gamma} \Sigma_{h,xx}^{-1} \left[ E_{h,t}(x_{t+1}) + \frac{1}{2} Var_{h,t}(x_{t+1}) + (1 - \gamma)\sigma_{h,1x} \right] + IHD_{h,t}.
\]

B. Evolution of trader types

Thus far, we derived the demand for assets of a given agent type but with no discussion on how the agents initially choose their types. In this section, we model the evolution of \( \eta_{ht} \), the fraction of agent type \( h \) at time \( t \). This is the so-called evolutionary part of the model and describes how beliefs about the best strategy are updated over time.

Following Brock and Hommes (1997, 1998), agents observe the past performance of each strategy and then decide between them. Agents have access to

\(^5\)A logarithmic investor has coefficient of risk aversion \( \gamma = 1 \); hence \( \theta = 0 \). Therefore, the portfolio rule of the investor is simply myopic, as we would expect. \( \theta = 0 \) sets the intertemporal hedging demand term to zero, and the only term left (that does not depend on \( \theta \)) is the myopic one.
fitness measures that are subjected to noise due to measurement errors or non-observable characteristics. The observed fitness of strategy $h$, $\bar{U}_{h,t}$, is given by

$$\bar{U}_{h,t} = U_{h,t} + \varepsilon_{h,t},$$

where $U_{h,t}$ is the deterministic part of the measure, and $\varepsilon_{h,t}$ an iid noise across types, drawn from a double exponential distribution. In this case, the probability that a given agent chooses strategy $h$ is given by the multinomial logit probabilities of a discrete choice when the number of agents tends to infinity. So, we describe the fractions $n_{ht}$ of trader types as follows:

$$\eta_{ht} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})},$$

where $U_{h,t-1}$ is the fitness measure of strategy $h$ evaluated in period $t - 1$, and $\beta$ is a parameter regulating the intensity of choice. The later is inversely proportional to the variance of the noise $\varepsilon_{h,t}$.

The measure of evolutionary fitness of strategy $h$ is the realised profits over a certain period, which is given by

$$U_{h,t} = (x_{t}) \alpha_{h,t} + \omega U_{h,t-1},$$

where $\omega$ is a memory parameter that reflects how slowly agents discount the success of past strategies when selecting their trading rules and $\bullet$ is the direct product operator. We then consider the simplest case: no memory, i.e., $\omega = 0$. In this case, (25) becomes

$$U_{h,t} = (x_{t}) \alpha_{h,t}.$$

C. The equilibrium

Assuming the market is in equilibrium, i.e., total demand, $\alpha_{t}^{d}$, equals total supply, $\alpha_{t}^{s}$, for each asset, the following equation holds true:

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6This ensures that $\beta = 0$ when the variance of noise is infinity. In this case, agents cannot observe differences in fitness and are not sensitive to differences in the performance of strategies. The other extreme situation is when the performances of the strategies could be perfectly observed, or $\beta = \infty$. In this case, all agents switch strategies when they see any difference in relative performances.
where the vector $\eta_{ht}$ denotes the (possibly different) fraction of trader type $h$ at date $t$ in each of the asset markets while considering $H$ different trader types.

Now, combining (22) and (27) for the case of zero outside supply shares (i.e., $\sigma_{t}^{d} = 0$) yields the market-clearing condition that closes the model:

$$\sum_{h=1}^{H} \eta_{ht} \cdot \alpha_{ht} = \alpha_{t}^{d} = \alpha_{t}^{s}$$

II. Empirical application

In this section, I assume the perspective of investors in the U.S.A. who would like to diversify their portfolio using the international stock markets. These investors classify strategies into fundamentalists and chartists. In each period, they need to decide whether to use the forecast given by one or the other model (i.e., choose their types).

If they believe that the market participants are very risk averse, then their assumptions about their investment horizons are extremely important. In this case the intertemporal hedging demand term dominates the demand for assets of long term investors. Considering only its myopic component to compute the relative performance of the strategies often generates different results.

I estimate the model assuming that investors can allocate funds between four major stock markets: U.S. (Dow Jones Industrials), UK (FTSE all share), Japan (Nikkei 500) and Hong Kong (Hang Seng).

The main objective of this exercise is to evaluate the impact of considering only the myopic component of the demand for assets on the results of the model. I show that the intertemporal hedging demand term is not only significant, but it dominates for very risk averse agents. I also show that the inclusion of the intertemporal hedging term in the demand for assets has different effects for fundamentalist and chartist agents. These effects also depend on their risk aversion. In addition, I show that the level of noise in the observed performances also has different impacts on the model results whether we consider the complete intertemporal demand for assets or only its myopic component.

Finally, I show that the proportion of trader types fluctuates according to the market conditions. These fluctuations are relatively more prominent for the Nikkei and Hang Seng markets. One explanation is that the two markets show clearer regime switches during the observed period.
### A. Data Description

I use quarterly data from the U.S., UK, Japan and Hong Kong stock markets. Table 1 reports the main descriptive statistics, all returns are in U.S. dollars. I estimate the fundamentalist and chartist models using the complete data set for each individual market. These data sets go until the first quarter of 2007, but they start at different dates. The Dow Jones starts at the second quarter of 1978; the FTSE starts at the first quarter of 1965; the Hang Seng starts at the third quarter of 1973; and the Nikkei starts at the first quarter of 1992.

For the estimation of the intertemporal asset allocation problem, however, I restrict attention to the common sample ranging from the first quarter of 1993 until the first quarter of 2007. This is the period when forecasts of these models exist.

Datastream is the source for the index values and dividend-price ratios. Quarterly data regarding the American consumption-wealth ratio comes from the Martin Lettau’s website\(^7\) and corresponds to the updated data set in Ludvigson and Lettau (2004), whereas the CPI series comes from the U.S. Department of Labor Statistics.

### B. Estimation

I construct the real stock return using the difference between the return on the stock index of each country and the U.S. inflation in the same period using the CPI. I report the results for $\Psi = 0.98$, $\beta = 10$ and $\gamma = 5$ or $\gamma = 50$. However, the model estimates for $\beta = \{0.25, 0.75, 0.5, 1, 5, 10, 20\}$ and $\gamma = \{1, 2, 5, 20, 50\}$ have the same qualitative results.

I restrict attention to a simple version of the model with two agents and four assets. I find the proportions of fundamentalists and chartists in two steps. First,

\(^7\)http://faculty.haas.berkeley.edu/lettau/

<table>
<thead>
<tr>
<th>Returns</th>
<th>Dow</th>
<th>FTSE</th>
<th>Nikkei</th>
<th>Hang Seng</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0085</td>
<td>0.0070</td>
<td>-0.0004</td>
<td>0.0178</td>
</tr>
<tr>
<td>Median</td>
<td>0.0172</td>
<td>0.0150</td>
<td>-0.0001</td>
<td>0.0437</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.1966</td>
<td>0.5763</td>
<td>0.2427</td>
<td>0.5158</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.3211</td>
<td>-0.3493</td>
<td>-0.3737</td>
<td>-0.7288</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.08</td>
<td>0.11</td>
<td>0.13</td>
<td>0.19</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.84</td>
<td>0.34</td>
<td>-0.46</td>
<td>-0.83</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>5.1</td>
<td>7.2</td>
<td>3.2</td>
<td>5.6</td>
</tr>
<tr>
<td>Observations</td>
<td>226</td>
<td>180</td>
<td>85</td>
<td>170</td>
</tr>
</tbody>
</table>

Table 1—Descriptive statistics for the series of Dow Jones Industrials, FTSE all shares, Nikkei 500 and Hang Seng real quarterly returns in US dollars
I determine their demand for assets as in (20). Next, I use this as an input to
determine the corresponding proportion of types given by (24). I use the constant
conditional correlation GARCH specification proposed by Bollerslev (1990) to
estimate the conditional variances and covariances in (20).

**Estimated agents’ models.** — Fundamentalist agents predict the real return on
every asset using the past dividend-price ratio:

\[ x_{t+1} = \mu + \rho_0 x_t + \rho_1 DP_t + \rho_2 DP_{t-l_2} + \rho_3 DP_{t-l_3} + \epsilon_t. \]  

Past real return \( x_t \) is included to eliminate serial correlation in the equation;
\( \epsilon_t \) is an error term; \( l_2 \) and \( l_3 \) are lags that vary according to the asset that agents
are forecasting. I choose the lags empirically to match the data.

Chartist traders use only past returns to forecast future returns for each asset. This
model is given by

\[ x_{t+1} = \mu + \rho_0 x_{t-l_0} + \rho_1 x_{t-l_1} + \epsilon_t. \]

I choose the lags empirically but the aim is to keep these lags small, given that
momentum is mostly a short term effect.

I estimate these models for each one of the \( n = 4 \) assets. They provide the
inputs for the (restricted) VAR that agents use to describe the market. Agents
estimate the parameters in (29) and (30) recursively, based on the information
available on each date. For example, agents use the information available up to
third quarter of 1999 to estimate \( \rho \) in fourth quarter of 1999.

Table 2 displays the results of these estimations for the two agent types in
each market (using the whole data set in the estimation). The fundamentalist
models fit the data much better than the chartist ones. The positive coefficients
of the lagged returns in the chartist models, however, are in accordance with the
previous findings of momentum effects. The overall positive coefficients of the
dividend-price ratios in the fundamentalist models are also in accordance with
the literature. The biggest difference among the markets is that we can find a
relationship between future return and the dividend-price ratio at much shorter
horizons for the Nikkei index. The shorter estimation sample (first quarter of
1991 until the first quarter of 2007) for the Nikkei does not allow to test if there
is a stronger relationship at longer horizons.

Table 3 reports the coefficient estimates of the \( GARCH(2,1) \) specification,
given in (31):

\[ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \beta \sigma_{t-1}^2 \]
Table 2—the table shows the estimated models of the two agent types in each market. Fundamentalist agents estimate for each market a model of the form: \( x_{t+1} = \mu + \rho_0 x_t + \rho_1 DP_t + \rho_2 DP_{t-3} + \rho_3 DP_{t-9} + \epsilon_t \), where \( x_t \) is the real return in time \( t \); \( DP_t \) is the dividend-price ratio in time \( t \); \( \epsilon_t \) is an error term and \( \mu \) and the \( \rho \) are the coefficients to be estimated. Chartist agents estimate for each market a model of the form: \( x_{t+1} = \mu + \rho_0 x_{t-1} + \rho_1 x_{t-1} + \epsilon_t \). In the table, the number in parenthesis gives the number of lags of the corresponding variable.
Table 3—Considering the GARCH (2, 1) given by $\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \alpha_2 \sigma_{t-2}^2 + \beta \epsilon_{t-1}^2$, the table shows the estimated coefficients $\alpha_1, \alpha_2$ and $\beta$ in each market. It also shows the sum of these coefficients.

<table>
<thead>
<tr>
<th></th>
<th>Dow Jones</th>
<th>FTSE</th>
<th>Hang Seng</th>
<th>Nikkei</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Chartist</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alpha-1</td>
<td>0.14</td>
<td>0.00</td>
<td>0.12</td>
<td>0.29</td>
</tr>
<tr>
<td>Alpha-2</td>
<td>0.00</td>
<td>0.25</td>
<td>0.78</td>
<td>0.05</td>
</tr>
<tr>
<td>Beta</td>
<td>0.79</td>
<td>0.61</td>
<td>0.10</td>
<td>0.36</td>
</tr>
<tr>
<td>Sum</td>
<td>0.92</td>
<td>0.86</td>
<td>1.00</td>
<td>0.69</td>
</tr>
<tr>
<td><strong>Fundamentalist</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alpha-1</td>
<td>0.18</td>
<td>0.00</td>
<td>0.06</td>
<td>0.04</td>
</tr>
<tr>
<td>Alpha-2</td>
<td>0.57</td>
<td>0.12</td>
<td>0.32</td>
<td>0.00</td>
</tr>
<tr>
<td>Beta</td>
<td>0.20</td>
<td>0.59</td>
<td>0.27</td>
<td>0.38</td>
</tr>
<tr>
<td>Sum</td>
<td>0.75</td>
<td>0.70</td>
<td>0.65</td>
<td>0.37</td>
</tr>
</tbody>
</table>

These univariate GARCH are inputs for the constant conditional correlation GARCH framework of Bollerslev (1990). The sum of the coefficients in the GARCH models are at most one. This indicates that they are all weakly stationary, though not necessarily with finite unconditional variance.

C. Model results

The components of the demand for assets. — Figures 1 and 2 show the relative importance of each component in the demand for assets of fundamentalist and chartist agents with two different levels of risk aversion. Equation (20) describes how to obtain the myopic and the intertemporal hedging demand terms in each graph.

Comparing the two columns in Figures 1 and 2, we see that as agents become more risk averse, the importance of the intertemporal hedging demand in relation to the myopic demand for assets increases. For very risk averse individuals and depending on the asset, the intertemporal hedging term in fact becomes the dominant component in the demand for assets. Figure 1 reveals that this happens with the Dow Jones and the FTSE for the fundamentalists whereas Figure 2 shows
Figure 1. The figure shows the relative importance of the myopic and the intertemporal hedging terms in the demand for assets of fundamentalist agents. The results correspond to the four markets: U.S. (Dow Jones), Japan (Nikkei), U.K. (FTSE) and Hong Kong (Hang Seng) and to a coefficient of relative risk aversion $\gamma = 5$ or $\gamma = 50$.

that this applies for every stock market index, but the Hang Seng, in the case of the chartist agents.

There are two main reasons that explain why the increase in risk aversion leads to an increase in the relative importance of the intertemporal hedging demand. The first is that it decreases the overall demand for risky assets, reducing the myopic demand term. The second is that the agents become more willing to hedge against changes in the investment opportunity set. Therefore, they demand more of assets with such properties (via the intertemporal hedging demand term).

Examining the pairs of Figures 2 and 1, we see that the intertemporal hedging demand term has different effects on the total demand for assets of fundamentalist and chartist agents. For instance, the intertemporal hedging demand for the Hang Seng is positive for the chartist agents, and negative for the fundamentalists.

The intertemporal hedging demand term with multiple assets. — The four assets present desirable intertemporal hedging characteristics given the negative covariance between shocks on their returns and shocks on the consumption-wealth
Figure 2. The figure shows the relative importance of the myopic and the intertemporal hedging components in the demand for assets of chartist agents. The results correspond to the four markets: U.S. (Dow Jones), Japan (Nikkei), U.K. (FTSE) and Hong Kong (Hang Seng) and to a coefficient of relative risk aversion $\gamma = 5$ or $\gamma = 50$. 
ratio. Therefore, in a single asset framework, the intertemporal hedging demand would be positive for all of them. In a multiple asset framework, however, the results can be different.

Fundamentalists have a negative intertemporal hedging demand for the Hang Seng because from a portfolio perspective this asset is very risky. Table 4 shows that the shocks on the return on the Hang Seng have the largest variance and covariance with the shocks on the returns on the other assets. This happens for both fundamentalist and chartist agents. However, fundamentalists have an overall larger intertemporal hedging demand. We can find the intuition for this result in Campbell, Chan and Viceira (2003). They note that the predictability of returns increases the demand for intertemporal hedge. As mentioned before, fundamentalist agents use models that predict returns more accurately than chartists. Therefore, keeping everything else constant, fundamentalists should have higher intertemporal hedging demands. As a consequence, they short the Hang Seng index to reduce the risk of their overall portfolio.

<table>
<thead>
<tr>
<th>Covariances (×10^{-3})</th>
<th>Dow Jones</th>
<th>FTSE</th>
<th>Hang Seng</th>
<th>Nikkei</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dow Jones</td>
<td>6.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FTSE</td>
<td>5.5</td>
<td>7.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hang Seng</td>
<td>8.1</td>
<td>8.4</td>
<td>21.1</td>
<td></td>
</tr>
<tr>
<td>Nikkei</td>
<td>4.2</td>
<td>5.0</td>
<td>6.3</td>
<td>16.2</td>
</tr>
</tbody>
</table>

Fundamentalist agents

<table>
<thead>
<tr>
<th>Covariances (×10^{-3})</th>
<th>Dow Jones</th>
<th>FTSE</th>
<th>Hang Seng</th>
<th>Nikkei</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dow Jones</td>
<td>4.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FTSE</td>
<td>3.6</td>
<td>5.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hang Seng</td>
<td>4.7</td>
<td>5.6</td>
<td>15.8</td>
<td></td>
</tr>
<tr>
<td>Nikkei</td>
<td>2.6</td>
<td>2.6</td>
<td>4.0</td>
<td>10.3</td>
</tr>
</tbody>
</table>

Table 4—Variance-Covariance matrix of the shocks on the expected returns estimated by chartist and by fundamentalist agents.

Estimated proportions of types. — Figure 3 shows the estimated proportions of fundamentalists given two different levels of risk aversion, $\gamma = 5$ and $\gamma = 50$. It compares the proportions obtained from the complete intertemporal demand for assets (i.e. including also the intertemporal hedging demand term) with the ones obtained from its myopic component alone.

Although not reported, the estimated excess covariance between shocks on the asset’s return and shocks on the consumption-wealth ratio, as given in equation (21), is negative. For $\Psi = 0.98 < 1$ and for an agent that is more risk averse than a logarithmic one (i.e., $\gamma > 1$), it would be possible to obtain a negative value for the intertemporal hedging demand term if the excess covariance between shocks on the asset’s return and shocks on the consumption-wealth ratio was positive.
Figure 3. The figure shows the proportions of fundamentalists in each market estimated from the complete intertemporal demand for assets and also from its myopic component alone. I assume a coefficient of relative risk aversion $\gamma = 5$ or $\gamma = 50$.

As expected, considering only the myopic component or the complete demand for assets results in significantly different estimations when the agents are very risk averse, i.e., $\gamma = 50$. When agents are not extremely risk averse, i.e., $\gamma = 5$, the estimated proportions do not change much from one formulation to another in the data set used here. This happens regardless of the fact that the intertemporal hedging demand term, shown earlier, is significantly large for agents with both levels of risk aversion.

Changing the intensity of choice $\beta$. — Figure 4 shows how the estimated proportions of agents change with the noise in the observed performances (captured by the values of $\beta$) given the myopic or intertemporal framework used. The plot shows that changing the value of $\beta$ affects the variation in the proportions of agents. The intensity of choice, $\beta$, is negatively correlated with the magnitude of the noise in the observed performance of the strategy. In other words, a high value of $\beta$ corresponds to a situation in which traders observe relative differences in performance more clearly. It increases the likelihood of the traders changing their types. This in turn results in a higher variation over time in the proportions of fundamentalists, as we see in the graph.

The picture also shows that this effect is stronger when we consider the com-
Figure 4. The graph shows the estimated proportions in the Dow Jones for $\beta = \{1, 5, 10\}$ considering the complete demand for assets or only its myopic component for $\gamma = 50$. It shows the relationship between the estimated proportions, the demand for assets and the different levels of noise in the observed performances (captured by the values of $\beta$).

The proportions of fundamentalists and the markets. — Figure 5 shows the variation in the proportions of fundamentalist traders according to the market conditions. It shows these variations in the four different markets plotting each index level (in US$) with the corresponding fundamentalist proportion. In common, the plots show a pattern of a decrease in the fraction of fundamentalists being followed by a reversal in prices and a subsequent increase in the fraction of fundamentalists. This pattern is clearer in the Hang Seng index, or during the period between the last quarter of 1998 and the last quarter of 2002 in the Nikkei and also, to a lesser extent, in the FTSE.

The decrease in the fraction of fundamentalists occurs because fundamentalist strategies are not successful in forecasting returns when prices do not follow the fundamentals. This is what happens between 1998 and 1999 especially in the
Figure 5. The figure shows the proportions of fundamentalists and the corresponding stock market index value when agents have a coefficient of relative risk aversion $\gamma = 50$.

Hang Seng and Nikkei indices.

When prices start to revert to the fundamentals, the traders begin to believe that fundamentalist strategies are correct. Subsequently, the proportion of fundamentalists increases until the first half of 2000, when the market prices are back to the level that they were in 1998. In the Hang Seng (and to a lesser extent in the FTSE also), we do not observe a reversal, but we see that the last increase in prices starting in the second half of 2003 is not consistent with fundamentals, as provided by our model. Finally, the participation of fundamentalists in the Dow Jones does not oscillate much.

III. Concluding remarks

In this paper I develop a new asset-pricing model in which agents with long investment horizons maximise a recursive utility function and choose the strategy used to forecast returns based on previous profitability. The model keeps many characteristics of earlier asset pricing models with heterogeneous beliefs. For instance, it has the ability to generate changes in prices that are not driven by fundamentals without requiring restrictive assumptions about the agent’s preferences or rationality.
The paper extends the literature on heterogeneous beliefs into two different directions. First by considering agents with long-term investment horizons as opposed to myopic investors. The empirical exercise shows that the component on the demand for assets that is ignored in a myopic framework can be significantly large. This is especially true when agents are very risk averse. In addition, the impact of changes in the parameters of the model is also different whether we consider the complete intertemporal asset demand or only its myopic component.

The paper also extends the literature on heterogeneous beliefs by considering an arbitrary large number of assets, \( n \). The negative intertemporal hedging demand for the Hang Seng by fundamentalist agents, for instance, would be positive in a single risky asset formulation given its desirable hedging properties.

REFERENCES


such as the noise in observed performances, captured by \( \beta \), or the level of risk aversion, \( \gamma \).


**Appendix**

**A1. Excess returns and the stochastic discount factor**

The basic equation of asset pricing (in terms of returns) can be written as follows:

\[ 1 = E_t [M_{t+1}(1 + R_{i,t+1})], \]

where \(M_{t+1}\) is the "stochastic discount factor" (SDF) that prices any asset in the economy. Equation (A1) can be developed into:
\[ 1 = E_t [M_{t+1} (1 + R_{i,t+1})] \]
\[ = E_t [M_{t+1}] \cdot E_t [(1 + R_{i,t+1})] + Cov_t (M_{t+1}, (1 + R_{i,t+1})) \]
\[ = E_t [M_{t+1}] \cdot E_t [(1 + R_{i,t+1})] + Cov_t (M_{t+1}, R_{i,t+1}). \]

Using the fact that \( E_t [M_{t+1}]^{-1} = (1 + R_{f,t+1}) \), obtained from equation (A1) for the risk-free asset, we have:

\[
E_t [(1 + R_{i,t+1})] = \frac{1 - Cov_t (M_{t+1}, R_{i,t+1})}{E_t [M_{t+1}]} = (1 + R_{f,t+1}) - \frac{Cov_t (M_{t+1}, R_{i,t+1})}{E_t [M_{t+1}]},
\]

and finally, the expression for the excess returns rearranging the equation once again is given by:

\[
E_t [(1 + R_{i,t+1}) - (1 + R_{f,t+1})] = - \frac{Cov_t (M_{t+1}, R_{i,t+1})}{E_t [M_{t+1}]},
\]

\[
E_t [R_{i,t+1} - R_{f,t+1}] = - \frac{Cov_t (M_{t+1}, R_{i,t+1})}{E_t [M_{t+1}]}.
\]

During the derivation of the approximate solution in the text, we obtain equation (13), that is similar to equation (A2). The left-hand side of equation (A2) is the expected excess return for asset \( i \). This expectation is given by the beliefs of the agents, modeled by the VAR described in the text. Agents with heterogeneous beliefs have different expectations of returns and therefore different demands for assets.

Note that, in the Euler equation (2), we obtain:

\[
M_{t+1} = \left\{ \delta \left( \frac{C_{h,t+1}}{C_{h,t}} \right) \right\}^\theta (1 + R_{h,p,t+1})^{-(1-\theta)}.
\]

In this case, the right hand side of equation (A2) depends on the agent’s choices, i.e., the covariance term depends on the portfolio composition, \( \alpha_{h,t}^* \) (via \( R_{h,p,t+1} \)) and consumption, \( C_{h,t}^* \), the two variables that the agent chooses and also the only two sources of variability in the SDF.

So, given the expectation on the left-hand side of equation (A2) and a consumption policy, we are able to determine the portfolio choice (i.e., asset demands).