Long Range Dependence in the Intraday Trading Data and Risk Neutral Market

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Overview

Portfolio Optimization: Markowitz Model
Derivative Pricing: Black Scholes Model
Risk management: RiskMetrics™ Method
- Multivariate Gaussian Distribution
- Brownian Motion
- Stationary and Markovian

Market Crash (Black-Monday, U.S. Financial Crisis 2008)

Fat-tailed Asymmetric Distribution
Stationary + Non-Gaussian
- Alpha Stable Model [1]
- Jump Diffusion Model [8]
- Variance Gamma Model [2]
- Tempered Stable Model [4, 5]
- Normal Tempered Stable [3, 11]
- Modified Tempered Stable [6]

Stochastic Volatility
Non-Stationary + Gaussian
- ARMA-GARCH model [10]
- Duan’s GARCH(1,1) model [12]
- Heston’s Stochastic Volatility Model [7]

Long Range Dependence (LRD)
Fractional Brownian Motion [13]
Fractional alpha stable process [14, 15]

Stochastic Volatility Model with Fattailed Asymmetric distribution
Non-Stationary + Non-Gaussian
- Alpha Stable GARCH model [1]
- Stochastic Volatility Levy Model [9]
- GARCH Model with Jumps
- GARCH Model with Tempered Stable Innov.

Stochastic Volatility + Fattailed Asymmetric Distribution + LRD
Overview

The propose of this paper is to find the answer to the following two questions:

- Is there long-range dependence in the risk-neutral asset return process?
- How can we detect the long-range dependence in the risk-neutral process including jumps?

To find the answer, the fractional Lévy process model will be discussed in this paper with an empirical calibration of the model to the market option price data.
Tempered Stable Subordinator

- \( \alpha \in (0, 2), \theta > 0 \)
- The pure jump Lévy process \( \tau = (\tau(t))_{t \geq 0} \) whose characteristic function \( \phi_{\tau(t)} \) is equal to

\[
\phi_{\tau(t)}(u) = \exp \left( -\frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha} t \left( (\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}} \right) \right). \tag{1}
\]

is a subordinator and referred to as the tempered stable subordinator with parameters \((\alpha, \theta)\).

Figure: Simulated sample path of the tempered stable subordinator \( \tau \) with parameters \( \alpha = 1.2 \) and \( \theta = 0.8 \).
Normal Tempered Stable (NTS) Process

Consider the Brownian motion $B = (B(t))_{t \geq 0}$ independent of $\tau$. Define $Z = (Z(t))_{t \geq 0}$ by the time-changed Brownian motion as

$$Z(t) = \beta(\tau(t) - t) + \gamma B(\tau(t)).$$

(2)

Then the process $Z$ is referred to as the $NTS$ process with parameters $(\alpha, \theta, \beta, \gamma)$ and denoted by $Z \sim NTS(\alpha, \theta, \beta, \gamma)$.

The mean of $Z(t)$ is equal to zero and the variance of $Z(t)$ is equal to

$$\text{Var}(Z(t)) = t \left( \gamma^2 + \beta^2 \left( \frac{2 - \alpha}{2\theta} \right) \right).$$

(3)

If $\gamma = \sqrt{1 - \beta^2 \left( \frac{2 - \alpha}{2\theta} \right)}$ and $|\beta| < \sqrt{\frac{2\theta}{2 - \alpha}}$ then $\text{Var}(Z(t)) = t$. In this case the process $Z$ is referred to as the $standard NTS$ process with parameters $(\alpha, \theta, \beta)$ and denoted by $Z \sim \text{stdNTS}(\alpha, \theta, \beta)$. 
fractional NTS (fNTS) Process

Let $K_H(t, s)$ be the Volterra kernel (See Appendix) and $Z \sim \text{stdNTS}(\alpha, \theta, \beta)$. The value $H$ is referred to as the Hurst Index.

The fractional standard NTS (fstdNTS) process $X = (X(t))_{t \geq 0}$ generated by $Z$ is defined by

$$
X(t) = \int_0^t K_H(t, s) dZ(s)
$$

$$
= \lim_{||P|| \to 0} \sum_{j=1}^{M} K_H(t, t_{j-1}) (Z(t_j) - Z(t_{j-1}))
$$

in distribution sense, where

$$
P : 0 = t_0 < t_1 < \cdots < t_M = t
$$

$$
||P|| = \max \{ \Delta t_j = t_j - t_{j-1} : j = 1, 2, \cdots, M \}.
$$

Univariate fractional tempered stable process was defined by the stochastic integral for the Volterra kernel in Houdre and Kawai (2006) based on subclasses of Rosinski’s tempered stable processes (Rosinski (2007)).

Figure: Simulated sample paths of NTS and fNTS processes with parameters $\alpha = 1.2$, $\theta = 0.36$, $\beta = -0.26$, and $H = 0.65$. 
the characteristic function of $X(t)$ is given by

$$
\phi_{X(t)}(u) = \exp\left( -\beta ui \int_0^t K_H(t, s) ds + \frac{2t\theta}{\alpha} 
- \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha} \int_0^t \left( \theta - i\beta u K_H(t, s) + \left( 1 - \beta^2 \left( \frac{2 - \alpha}{2\theta} \right) \right) \frac{u^2(K_H(t, s))^2}{2} \right)^{\frac{\alpha}{2}} ds \right). 
$$

For $n \in \{1, 2, \cdots, N\}$, the covariance between $X(s)$ and $X(t)$ is equal to

$$
\text{Cov}(X(s), X(t)) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad s, t > 0. \tag{4}
$$
For a given stochastic process $Y = (Y(t))_{t \geq 0}$, the summation

$$\sum_{j=1}^{\infty} E[(Y(1) - Y(0))(Y(j + 1) - Y(j))]$$

diverges, then we say that $Y$ exhibits long-range dependence. By L’Hopital’s rule, we have

$$E[X(1)(X(j + 1) - X(j))]$$

$$= \frac{v}{2} ((j + 1)^{2H} - 2j^{2H} + (j - 1)^{2H})$$

$$= \frac{v}{2} j^{2H-2} \left( j^2 \left( \left( 1 + \frac{1}{j} \right)^{2H} - 2 + \left( 1 - \frac{1}{j} \right)^{2H} \right) \right)$$

$$\rightarrow vH(2H - 1)j^{2H-2} \text{ as } j \rightarrow \infty,$$

where $v = \left( \gamma^2 + \beta^2 \left( \frac{2-\alpha}{2\theta} \right) \right)$. Hence, $\sum_{j=1}^{\infty} E[X(1)(X(j + 1) - X(j))]$ diverges, i.e. the process $(X(t))_{t \geq 0}$ has long-range dependence, when $\frac{1}{2} < H < 1$. 

Aaron Kim
LRD in Risk Neutral Market
March 19, 2015 10 / 32
Let $T$ be a time horizon or option maturity time, and consider the discrete time steps such that

$$\xi = t_0 < t_1 < \cdots < t_M = T$$

for fixed large positive integer $M$, where $0 < \xi \ll 1$. 
A. The statistical price process

We assume that the statistical price process of the underlying stock is

\[ S(t_m) = S(t_0) \exp(\mu t_m - w(t_m) + \hat{X}(t_m)), \quad m = 1, 2, \ldots, M, \]

where \( \mu \in \mathbb{R} \) is the expected return, \( w(t_m) = \log E[\exp(\hat{X}(t_m))] \), and \((\hat{X}(t_m))_{m=0,1,\ldots,M}\) is an approximation of the process \(fNTS(H, \alpha, \theta, \beta, \gamma)\) such that

\[ \hat{X}(t_m) = \sum_{j=1}^{m} K_H(t_m, t_{j-1}) (Z(t_j) - Z(t_{j-1})) , \quad (5) \]

with \( \hat{X}(t_0) = 0 \) and \( Z \sim NTS(\alpha, \theta, \beta, \gamma) \).
B. The risk-neutral price process

The risk-neutral price process is assumed to be

\[ S(t_m) = S(t_0) \exp(rt_m - \tilde{w}(t_m) + \tilde{X}(t_m)), \quad m = 1, 2, \cdots, M, \]

where \( r > 0 \) is the risk-free interest rate, \( \tilde{w}(t_m) = \log E[\exp(\tilde{X}(t_m))] \) and \( (\tilde{X}(t_m))_{m=1,2,\cdots,M} \) is an approximation of \( f\text{NTS}(\tilde{H}, \tilde{\alpha}, \tilde{\theta}, \tilde{\beta}, \tilde{\gamma}) \) such that

\[ \tilde{X}(t_m) = \sum_{j=1}^{m} K_{\tilde{H}}(t_m, t_{j-1}) \left( \tilde{Z}(t_j) - \tilde{Z}(t_{j-1}) \right), \quad \tilde{X}(t_0) = 0 \]

with \( \tilde{X}(t_0) = 0 \) and \( \tilde{Z} \sim NTS(\tilde{\alpha}, \tilde{\theta}, \tilde{\beta}, \tilde{\gamma}) \).
Suppose $\mathbb{P}$ and $\mathbb{Q}$ are measures generated by two finite discrete processes $\hat{X} = (\hat{X}(t_0), \hat{X}(t_1), \cdots \hat{X}(t_M))$ and $\tilde{X} = (\tilde{X}(t_0), \tilde{X}(t_1), \cdots \tilde{X}(t_M))$, respectively. Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent since the distributions of $\hat{X}$ and $\tilde{X}$ have the same domain.

We can say that the statistical and risk-neutral price processes are well defined when

$$K_H(t_m, t_0) \leq \frac{1}{\gamma^2} \left( \beta + \sqrt{\beta^2 + 2\gamma^2 \theta} \right)$$

(6)

and

$$K_{\tilde{H}}(t_m, t_0) \leq \frac{1}{\tilde{\gamma}^2} \left( \tilde{\beta} + \sqrt{\tilde{\beta}^2 + 2\tilde{\gamma}^2 \tilde{\theta}} \right)$$

(7)

for all $m = 1, 2, \cdots, M$

Motivation: We would consider the RDTS process instead of the NTS process to escape it.
The put price under the risk-neutral price process can be calculated by applying the Fast Fourier Transform method as in the following proposition:

**Proposition**

Consider a put option with the maturity $T$ and the strike price $K$. Let

$$
\phi(\zeta) = E \left[ \exp \left( i\zeta \left( rT - \tilde{w}(T) + \tilde{X}(T) \right) \right) \right].
$$

If (6) and (7) hold for all $m = 1, 2, \cdots, M$ and there is $\rho > 0$ such that $\phi(u + i\rho) < \infty$ for all $u \in \mathbb{R}$, then the put option price at time 0 is given by

$$
P = \frac{K^{1+\rho}e^{-rT}}{\pi S_0^\rho} \text{Re} \int_0^\infty e^{-iu \log(K/S_t)} \frac{\phi(u + i\rho)}{(\rho - iu)(1 + \rho - iu)} du.
$$

(8)
The parameters of the risk-neutral price process are calibrated to the option market prices using the least square error minimizing method.

The data investigated in this section is S&P 500 index put option market price on August 7, August 8, September 10, September 15 (the date of Lehman Brothers Collapse), and September 16, 2008.

Suppose the 13-week Treasury Bill Index is as the risk free rate of return, and set $\Delta t_j = 1/252$ and $\xi = 10^{-10}$.

In this calibration, fix the parameter $\tilde{H}$ and then estimate parameters $(\tilde{\alpha}, \tilde{\theta}, \tilde{\beta}, \tilde{\gamma})$ fit to the market put prices.

We repeat the calibration for each $H$ in $\{0.51, 0.52, \cdots, 0.70\}$ and find the parameter $\tilde{H}$ and corresponding parameters $(\tilde{\alpha}, \tilde{\theta}, \tilde{\beta}, \tilde{\gamma})$ that minimize the root mean square error (RMSE).
Figure: Put option prices at August 6
Figure: Put option prices at September 16, 2008.
## Model Calibration

### Table: Calibrated model parameters and errors

<table>
<thead>
<tr>
<th>Date(2008)</th>
<th>Model</th>
<th>$\tilde{H}$</th>
<th>$\tilde{\alpha}$</th>
<th>$\tilde{\theta}$</th>
<th>$\tilde{\beta}$</th>
<th>$\tilde{\gamma}$</th>
<th>ARPE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 6</td>
<td>NTS</td>
<td>0.5</td>
<td>1.9672</td>
<td>0.1650</td>
<td>−1.3218</td>
<td>0.1818</td>
<td>0.5008</td>
<td>1.1386</td>
</tr>
<tr>
<td></td>
<td>fNTS</td>
<td>0.53</td>
<td>1.9723</td>
<td>0.0001</td>
<td>−1.4536</td>
<td>0.2035</td>
<td>0.3918</td>
<td>0.9606</td>
</tr>
<tr>
<td>Aug. 7</td>
<td>NTS</td>
<td>0.5</td>
<td>1.7402</td>
<td>17.2187</td>
<td>−2.4980</td>
<td>0.0895</td>
<td>0.4865</td>
<td>1.3915</td>
</tr>
<tr>
<td></td>
<td>fNTS</td>
<td>0.53</td>
<td>1.8817</td>
<td>4.8188</td>
<td>−1.6757</td>
<td>0.1688</td>
<td>0.3364</td>
<td>0.9470</td>
</tr>
<tr>
<td>Sep. 10</td>
<td>NTS</td>
<td>0.5</td>
<td>1.1583</td>
<td>18.3355</td>
<td>−1.1881</td>
<td>0.1710</td>
<td>0.2244</td>
<td>1.3862</td>
</tr>
<tr>
<td></td>
<td>fNTS</td>
<td>0.76</td>
<td>0.8770</td>
<td>225.9969</td>
<td>−0.1679</td>
<td>0.0181</td>
<td>0.0855</td>
<td>0.6811</td>
</tr>
<tr>
<td>Sep. 15</td>
<td>NTS</td>
<td>0.5</td>
<td>0.8745</td>
<td>10.3562</td>
<td>−0.8041</td>
<td>0.1845</td>
<td>0.0829</td>
<td>3.4317</td>
</tr>
<tr>
<td></td>
<td>fNTS</td>
<td>0.69</td>
<td>1.9136</td>
<td>1.0703</td>
<td>−1.2442</td>
<td>0.0621</td>
<td>0.0724</td>
<td>1.5826</td>
</tr>
<tr>
<td>Sep. 16</td>
<td>NTS</td>
<td>0.5</td>
<td>0.4328</td>
<td>26.7165</td>
<td>−1.0298</td>
<td>0.1944</td>
<td>0.1082</td>
<td>3.4794</td>
</tr>
<tr>
<td></td>
<td>fNTS</td>
<td>0.69</td>
<td>1.9107</td>
<td>1.7434</td>
<td>−1.2347</td>
<td>0.0657</td>
<td>0.0529</td>
<td>1.2003</td>
</tr>
</tbody>
</table>
We can graphically see that the fNTS model fits to the market put price remarkably better than the NTS model on September 16, 2008, while the fNTS model is slightly better than the NTS model on August 6, 2008.

Based on ARPE and RMSE values in the Table 1, we can see that the fNTS option pricing model has smaller errors than the NTS option pricing model.

That means the fNTS model better explains the market risk-neutral price process than the NTS model in this investigation.

The risk-neutral Hurst index $\tilde{H}$ on September 10, 2008 is largest among the five cases in this investigation.

The $\tilde{H}$ values on September 10, 15 and 16, 2008 are larger than $\tilde{H}$ values on August 6 and 7, 2008.

Hence we can say that the long-range dependence in the risk-neutral price process for the volatile market around the Lehman Brothers Collapse was stronger than the case of one month before the Lehman Brothers Collapse.
The fractional NTS process is defined and applied to the option pricing model.

The fNTS option pricing model is developed based on the discrete finite process of an approximation of the fNTS process.

Using empirical study based on the fNTS model, one can detect the long-range dependence of the risk-neutral process.

By fNTS model calibration for the S&P 500 index option market, we observe the risk-neutral Hurst index $\tilde{H}$ is larger than 0.5, that is we capture the long-range dependence in the risk-neutral price process for S&P 500 index option.

The volatility smile effect and volatility term structure come from not only the fat-tail property but also the long range dependence.

We observe that the long-range dependence in the risk-neutral process for the volatile market around the Lehman Brothers Collapse is stronger than the case of one month before the Lehman Brothers Collapse.
Further Study 1: fractional non-stationary normal tempered stable model

- **Fractional Tempered Stable Subordinator**
  Consider the fractional Lévy process $T_{H_1} = (T_{H_1}(t))_{t \geq 0}$ defined by

  $$
  \tau_{H_1}(t) = \int_0^t K_{H_1}(t, u) d\tau(u),
  $$

  \hspace{1cm} (9)

  where $\tau$ is the tempered stable subordinator with parameter $(\alpha, \theta)$, and $K_{H_1}$ is the Volterra kernel. Then the process $T_{H_1}$ is referred to as *fractional tempered stable subordinator* with parameter $(H_1, \alpha, \theta)$.

- **Consider the fractional Brownian Motion** $B_{H_2} = (B_{H_2}(t))_{t \geq 0}$ defined by

  $$
  B_{H_2}(t) = \int_0^t K_{H_2}(t, u) dB(u),
  $$

  \hspace{1cm} (10)

  where $B$ is Brownian Motion, and $K_{H_2}$ is the Volterra kernel.
Further Study 1: fractional non-stationary normal tempered stable model

- Suppose that \((B_{H_1}(t))_{t \geq 0}\) and \((T_{H_2}(t))_{t \geq 0}\) are independent.
- We construct option pricing model in the risk-neutral space as

\[
S(t) = \frac{S(0) \exp(rt + \sigma X(t))}{E[\exp(\sigma X(t))]},
\]

where \(X = (X(t))_{t \geq 0}\) defined by

\[
X(t) = \beta(T_{H_1}(t))^{2H_2} + B_{H_2}(T_{H_1}(t)), \quad \beta \in \mathbb{R}. \tag{11}
\]

- Then we refer to \(X\) as the \textit{fractional non-stationary normal tempered stable} process.
- \((B_{H_2}(t))_{t \geq 0}\) is the Brownian motion if \(H_2 = \frac{1}{2}\)
- \((T_{H_1}(t))_{t \geq 0}\) is the tempered stable subordinator if \(H_1 = \frac{1}{2}\)

Further Study 2: fractional RDTS model

Instead of NTS process, we can construct fractional Lévy process with the RDTS process as

\[ X(t) = \int_0^t K_H(t, s)dZ(s) \]

where \( K_H(t, s) \) be the Volterra kernel and \( Z \sim \text{stdRDTS}(\alpha, \lambda_+, \lambda_-) \). Since RDTS distribution has Laplace transform for entire real number, we are free from (6) and (7) when we construct the option pricing model.
Further Study 3: Long Range Dependence Stochastic Volatility Lévy Process Model

Stochastic Volatility Tempered Stable Model

\[ S(t) = \frac{S(0) \exp(rt + X(t))}{E[\exp(X(t))]}, \]

where

- \( X(t) \sim CTS(\alpha, C(t), \lambda_+, \lambda_-, \mu) \)
- \( C(t) \) is a strictly positive fractional RDTS process given by

\[ C(t) = \sigma \int_0^t K_H(t, s)dZ(s), \quad \sigma > 0 \]

- \( Z(t) \) is the strictly positive fractional RDTS subordinator process.

The model has LRD, Fattails and jumps in Volatility process, Fattails and jumps in the driving process.
Volterra kernel $K_H : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, $H \in (0, 1)$:

$$K_H(t, s) = c_H \left( \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{-\frac{1}{2}} - \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}} (u - s)^{H - \frac{1}{2}} du \right) 1_{[0, t]}(s)$$

$$c_H = \left( \frac{H (1 - 2H) \Gamma\left( \frac{1}{2} - H \right)}{\Gamma(2 - 2H) \Gamma(H + \frac{1}{2})} \right)^{\frac{1}{2}}.$$
Appendix: Stochastic Integral

The $N$-dimensional fractional NTS (fNTS) process, $X = (X(t))_{t \geq 0}$ with $X(t) = (X_1(t), X_2(t), \cdots, X_N(t))^T$:

$$X_n(t) = \lim_{||P|| \to 0} \sum_{j=1}^{M} K_H(t, t_{j-1}) (Z_n(t_j) - Z_n(t_{j-1}))$$

in distribution sense for $n \in \{1, 2, \cdots, N\}$, where

$$P : 0 = t_0 < t_1 < \cdots < t_M = t$$

is a partition of the interval $[0, t]$ and

$$||P|| = \max\{t_j - t_{j-1}|j = 1, 2, \cdots, M\}.$$
Appendix: Series representation for the tempered stable subordinator

Series representation for the tempered stable subordinator $T$:

$$T(t) = \lim_{M \to \infty} \sum_{j=1}^{M} 1_{(0,t)}(\tau_j) \left( \left( \frac{\alpha \xi_j \Gamma \left(-\frac{\alpha}{2}\right)}{2\theta^{1-\frac{\alpha}{2}} T} \right)^{-\frac{2}{\alpha}} \wedge \frac{e_j u_j^{2\alpha}}{\theta} \right), \quad t \in [0, T], \quad (12)$$

where

- $\{u_j\}$ is an iid sequence of random variables on $(0, 1)$,
- $\{e_j\}$ and $\{e'_j\}$ are an iid sequence of exponential random variables with parameter 1,
- $\xi_j = e'_1 + e'_2 + \cdots + e'_j$,
- $\{\tau_j\}$ be an independent and identically distributed uniform random variable in $[0, T]$, where $T > 0$ is fixed.
- and assume that $\{u_j\}$, $\{e_j\}$, $\{e'_j\}$ and $\{\tau_i\}$ are independent.
The calibrated parameters are provided in Table 1. To evaluate the performance, the average relative percentage error (ARPE) is provided together with RMSE in the table. RMSE and ARPE are defined as follows:

\[
\text{RMSE} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \frac{(P_j - \hat{P}_j)^2}{N}},
\]

\[
\text{ARPE} = \frac{1}{N} \sum_{j=1}^{N} \frac{|P_j - \hat{P}_j|}{P_j}.
\]

where \(\hat{P}_j\) and \(P_j\) are model prices and observed market prices of put options with strikes \(K_j, j \in \{1, \ldots, N\}\), and \(N\) is the number of observed put option prices.
Rapidly Decreasing Tempered Stable (RDTS) Distribution

Characteristic function:

\[ \phi_X(u) = \exp (iun + C(G(iu; \alpha, \lambda_+) + G(-iu; \alpha, \lambda_-))) , \]

where \( \alpha \in (0, 2) \backslash \{1\} \), \( C, \lambda_+, \lambda_- > 0 \), \( m \in \mathbb{R} \), and

\[
G(x; \alpha, \lambda) = 2^{-\frac{\alpha}{2} - 1} \lambda^\alpha \Gamma \left( -\frac{\alpha}{2} \right) \left( M \left( -\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2} \right) - 1 \right) \\
+ 2^{-\frac{\alpha}{2} - \frac{1}{2}} \lambda^{\alpha - 1} x \Gamma \left( \frac{1 - \alpha}{2} \right) \left( M \left( \frac{1 - \alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2} \right) - 1 \right),
\]

and \( M \) is the confluent hypergeometric function.
Rapidly Decreasing Tempered Stable (RDTS) Distribution

Properties and Advantages:
- It describes skewness and leptokurtosis of empirical distribution.
- Easy to add the GARCH effect.
- Its jump intensity measure has thinner tails than the case of the CTS distribution.
- The characteristic function is given by analytic form.
- It has finite moments for all orders.
- The exponential moments are defined on entire real numbers. i.e. \( E[e^{\theta X}] < \infty \) for all \( \theta \in \mathbb{R} \). Hence, Duan’s GARCH model can be generalized to the RDTS-GARCH model without additional restriction.
- Estimating parameter with MLE is slower than the normal distribution.
- RDTS random numbers can be simulated by the method of Bianchi et.al(2008).
Rapidly Decreasing Tempered Stable (RDTS) Distribution

Disadvantages:

- Estimating parameter with MLE is slower than the normal distribution.
- Simulating RDTS random numbers is slower than simulating $\alpha$-stable random number.
- Its characteristic function is more complex than that of the CTS distribution.