Collusion under incomplete information on the discount factor

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Abstract

The gradual increase of prices at the start of collusion is a recurrent pattern that has been observed in many discovered cartels and in lab experiments. This paper seeks to capture this pattern in a model where firms have private information as to their respective discount rate. In a repeated Bertrand pricing duopoly game we show that, in order to maximize their payoffs while preventing low types from mimicking, patient firms adopt a price scheme which feature a transition phase. Prices gradually increase over time before reaching the highest sustainable price. We determine the best speed of price increase and the delay before which the best price level is attained. We also characterize the Payoff frontier for a class of strategies and exhibit an essential trade-off which produces a smooth transition phase.

Keywords: Collusion, incomplete information, transition phase

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1 Introduction

Identifying patterns in collusive practices allows antitrust and competition authorities to detect and prosecute illegal practices. A well-recognized pattern is the gradual increase of prices during cartel's formation, or at the start of collusive schemes.

A number of discovered cartels were quite explicit about gradually raising prices. Indeed, most cartel price paths show a gradual increase, as in citric acid and lysine, several of the vitamines, and graphite electrode[1]. This pattern is also observed in lab experiments. In Kujal, Harrington and Hernan-Gonzalez (2013), for symmetric duopoly treatment groups without communication, prices increase gradually before reaching a high supra-competitive level[2].

The fact that firms, in a (tacit or explicit) collusive agreement, start by coordinating on a gradual price increase, before reaching a high supra-competitive price seems prevalent. However, this "transition phase", is not captured by the standard models of collusion. In repeated games, optimal collusive schemes set the highest sustainable price from the first period on.

This research question is tackled indirectly in Harrington (2015). In his model, the goal is to capture how firms can coordinate on collusive schemes without express communication. What type of "mutual understanding" results in supra-competitive outcomes? More precisely, the main obstacle to collusion is not a sustainability issue, but rather an equilibrium selection problem. In this view, the analysis uses epistemology game theory concepts to represent firms behavior. In particular, firms have beliefs about what possible strategies their rivals are playing. Each action a firm takes, does not only affect firms' payoff but also affects rivals' beliefs about which strategy this firm employs. Under this framework, a collusive "equilibrium" is reached when each firm has correctly identified their rivals' strategy, and the price is supra-competitive. On a collusive path the fact that firms gradually increase their prices is interpreted as the fact that firm's beliefs are gradually narrowed down to a singleton. This is "mutual understanding": firms gradually realize that their rival are indeed playing a collusive strategy.

The current paper presents an alternative rationale for the gradual increase of prices at the start of a collusive scheme, using classic game theory concepts. We study collusion among firms that are privately informed as to their respective discount factors. In this model, we study Pareto optimal collusive schemes and show that they feature a "transition phase" : prices gradually increase before reaching a stable and high supra-competitive level.

To explain why under this assumption a transition phase emerges, one may consider an ideal collusive scheme where each firm is asked to play the highest sustainable price with respect to its discount rate. Thus, patient (high type) firms play high prices, and impatient (low type) firms play relatively low prices. In this ideal schedule, impatient firms are willing to mimic patient firms. The deviation profits at high prices obtained by an impatient firm mimicking exceed their anticipated low type collusive profit. To reduce the incentives to mimic, high type firms take advantage of the relative impatientness of low type firms by delaying the period in which the highest sustainable price level is reached. This delay is more harmful to impatient firms than to patient firms and thus ease type separation. In this sens the transition phase is viewed as a distortion in the mechanism design sens. The gradual increase of prices is interpreted as a costly variation from the "first-best" collusive scheme[3] that is necessary to sort types. It is a "second-best" price path that yields for patient firms Pareto payoffs given that it prevents impatient firms from mimicking.

The assumption that firms are privately informed on their discount factor is also analyzed in Harrington and Zhao (2012). In this model, they study a repeated version of the Prisoner's Dilemma, assuming one player is patient while the other has discount factor of 0 (myopic). The authors study a class of equilibria with delay, where patient player's type gradually elicit their rival's type. Hence, they show that delay (both in the play of "cooperation", and in type revelation) can be observed in equilibrium.

[1] See J. E. Harrigton, How do Cartel Operate? p.19

[2] See page 14 for the result of the baseline treatment group without communication.

[3] The "first-best" collusive scheme refers to the best collusive scheme in complete information. In which the highest sustainable price is played straight away.

Our analysis departs from their paper in the construction of the transition phase. The transition phase in our analysis refers to a gradualism in prices not in beliefs. In this sens, it is not a learning phase. It interprets the gradual increase of prices as a distortion of the patient firm's collusive path in order to induce early type separation. Additionally, price gradualism is the solution of a maximization problem. More precisely, increasing price path is solution to a problem which maximizes profits for patient firm while restricting mimicking profit from low type firm.

An other important part of the literature on collusion with incomplete information has focused on private information on firm's cost. Athey and Bagwell (2001), Athey, Bagwell and Sachirico (2004), Skryzpracz and Hopenhayn (2004), develop models where firms play a repeated Bertrand pricing game and have private information about their cost independently drawn each period. Athey and Bagwell (2008) analyze the case where cost type has some degree of persistency.

Their approach explains why one observes stable collusive prices overtime. More precisely, while the environment is constantly changing (such as privately observed cost shocks), collusive prices do not adapt to these shocks. In their analysis, best collusive schemes trade off productive efficiency (low cost type produces all outputs) with on-path distortions generated by truth-telling. In most cases, the second effect dominates and thus pooling equilibria are often optimal.

We use the same Bertrand pricing game, but we assume firm's costs are common knowledge, firms are privately informed on their discount factor. We believe the analysis presented shares common points with the paper of Kartal (2016). This paper presents a repeated principal-agent interaction with relational contract and private information on the principal's discount factor. It is shown that when patient principals signal their type the equilibrium path features a gradual increase in the agent's effort level. We claim that in a model of imperfect competition, the trade-off producing effort gradualism in Kartal (2016) would apply for the transition phase and enhance our result[4].

In the following analysis gradualism for the price level emerges as well, but from a different source. We characterize the optimal price path speed, and length, and also the Pareto frontier after separation. We exhibit that the transition phase is not "bang-bang". In the sens that the best transition phase does not fully deter low types in the start (by setting very low prices) and then jumps directly to the highest sustainable price for patient firms[5].

We begin our formal analysis by characterizing the best equilibrium payoffs under complete information, for firms with asymmetric discount factors. This question has been studied in Harrington (1988) and in Obara, Zincenko (2017). Using these results we present the equilibrium payoffs for given discount factors and characterize its Pareto frontier. The complete information frontier will be interpreted as first-best payoffs.

We then analyze the game under incomplete information. We show pooling equilibria are unappealing, and so we focus on separating equilibria. One way to sort firms' types is to let high type firms signal their types by taking action that low type firms cannot profitably replicate. Related to a money-burning idea, a patient firm can undergo losses in the first periods which impatient firms cannot afford since they value relatively more the present. A second way is based on a screening idea: give a rent to the type that is willing to mimic. Since low type firms are willing to mimic patient firms, patient firms can give them a rent in the first periods to induce revelation, and before playing the best collusive prices (maybe with delay) in the case their rival is patient. In both cases, the equilibrium path employed by patient firms must maximize their payoff while restricting as mush as possible what an impatient firm can obtain by mimicking. This defines the optimal use of instrument problem analyzed in section 5.

We then solve the optimal use of instruments problem. At the continuation equilibrium for patient firms, the price path should yield the best profits for high type firms while lowering deviating profits for mimicking firms, in order to ease separation. Following mechanism design semantic, as these prices are used to sort types, the whole path is understood as an "instrument". We will show that the solution feature a gradual increase of prices. We also characterize the Pareto frontier of this

[4] Discussed in more details in section 6.2.

[5] The transition phase will feature intermediary prices, and approach in a non-trivial manner the best collusive price. This is true despite the problem being linear (payoffs and constraints), and allowing for below cost pricing (which does not limit the scope for deterring low types).

problem, and present properties about the transition phase such as the speed of price increase and its length.

Next, we construct the Pareto frontier for the money-burning strategies. We explain why the price path is not "bang-bang" despite the problem being completely linear.

The paper proceeds as follows. Section 2 introduces the model. Section 3 analyzes the complete information game. In section 4 we comment the subclass of strategies and the parameter space considered. Section 5 and 6 analyze the optimal use of instrument problem and Pareto frontier of money-burning strategies. Section 7 concludes.

2 The Model

In this section we introduce the model. There are two firms A and B that meet in periods $t = 0, 1, ..., \infty$. Firms A and B engage every period in Bertrand competition for sales in a homogeneous-good market, with no costs. We assume that in each period, demand is inelastic, and there is a unit mass of identical consumers with the same reservation price r, where $r > 0$.

In a period t, firm A and B simultaneously select prices $p_{A,t}$ and $p_{B,t}$. The firm with the lowest price serves the market. It is typically assumed that market demand is equally allocated in case of tie. While there is no loss of generality in assuming this equal allocation rule when firms are symmetric, this specification may prove to be restrictive when firms are asymmetric. Henceforth, following Harrington (1988), we will allow firms to arbitrarily allocate market demand amongst themselves in case of tie, neglecting the strategic decision of an allocation. Let $\alpha_t \in [0, 1]$ be the market share of firm A in period t in case of tie.

Before period 0, firm B learns its discount rate which can be either $\delta_L \in (0, 1)$ or $\delta_H \in (0, 1)$ with $\delta_H > \delta_L$, firm's B discount rate is private. Firm A discounts the future at rate δ_H, which is common knowledge. Firm A has a prior $\rho \in (0, 1)$ on firm B's being high type. Throughout the analysis we call firm B(δ_L) if she is of the type δ_L, and B(δ_H) if she is of the type δ_H.

In the repeated game, a firm's strategy is a mapping the set of possible histories to the set of prices. And the payoff function is the sum of the discounted profits. We will focus on subgame perfect equilibrium (SPE), and more precisely on the set of payoffs achievable in SPE given the discount rate of both firms.

In the incomplete information game, the strategy for firm B is a mapping from the set of possible histories and types to the set of prices, and its payoff function is the sum of the discounted profits given B's type. While firm A's strategy is a mapping from the set of possible histories to the set of prices, and its payoff function is the expected sum of discounted profits given its prior ρ on firm B's type. We will focus on perfect Bayesian equilibria (PBE), and the set of payoffs achievable in PBE. In the analysis we focus on strategies which punish off-path deviations with infinite reversion to the stage game Nash equilibrium (NE). Hence, the continuation equilibrium after off-path deviation is PBE with any beliefs' specification since the type does not affect the per period profit.

2.1 Notations

In most of the equilibrium paths, firms will set the same prices, and these prices will be positive. For those we adopt a specific notation in terms of per-period profit.

Definition For a given equilibrium path $\{(p_t, \alpha_t)\}_{t \geq 0}$, let $\pi_{A,t}$ and $\pi_{B,t}$ be the per period profit of firm A and B:

$$\pi_{A,t} = \alpha_t p_t$$
$$\pi_{B,t} = (1 - \alpha_t) p_t$$

And let $\Pi_{k,t},\ k = A, B$ be the average continuation profit from period t on : $\Pi_{k,t} = (1-\delta_H)\sum_{s\geq t}\delta_H^{s-t}\pi_{k,t}$

Remarks

Because these notations will be used for positive price paths we have the following restrictions:

$$\pi_{A,t} + \pi_{B,t} \leq r$$
$$\pi_{A,t} \geq 0$$
$$\pi_{B,t} \geq 0$$

The same restrictions apply to $\Pi_{A,t}$ and $\Pi_{B,t}$, and will be maintained throughout the analysis.

In the next parts we characterize the best $(\Pi_{A,t}, \Pi_{B,t})$ achievable in equilibrium. For a given set of equilibrium payoffs, the best payoffs are located on its upper contour set, the Pareto frontier. The following definitions specify the notations.

Definition *Pareto frontier*
Let A be a set of $\mathbb{R}^2$, the Pareto frontier of A noted $F\{A\}$ is

$$F\{A\} = \{a \in A,\ |\ \nexists\ a' \in A,\ \text{s.t.}\ \ a' \geq a\ \ \text{and}\ \ b \neq a\}$$

The next definitions is required to set up Pareto frontier problems in a recursive form.

Definition Let A and B be two correspondences mapping $E \rightrightarrows E$. $\oplus$ is an operator which associates to two correspondences the set of

$$A \oplus B \overset{\text{def}}{=} \{a + b \mid (a, b) \in (A(b), B(a))\}$$

Remark that this operator is not the Minkowsky sum of the span of A and B. It only sums the fixed-points of the correspondence (A,B).

In the next two subsections, we describe strategies considered in the repeated game under complete information, and under incomplete information.

2.2 Strategies under complete information

The game under complete information may be interpreted as a benchmark. The best payoffs achievable in equilibrium may be interpreted as first best payoffs.

To characterize the best equilibrium payoffs it is enough to consider only grim-trigger strategies with positive symmetric prices[6]. In this perfect competition game, reverting to the Nash equilibrium for the reminder of the horizon yields the most severe punishment: each firm receives its minmax payoff of 0.

[6]Negative prices are trivially sub-optimal. Path with different prices are payoff equivalent to path with symmetric prices provided a careful choice of market share. One can show that different prices weakly increases the incentives to deviate compared to symmetric prices.

Definition *Grim-trigger strategy* Let $\{(p_t, \alpha_t)\}_{t\geq 0}$ be the targeted equilibrium path, of the Grim-trigger strategy called GT. We have:

$$\begin{array}{ll} \text{If } h_s = \{p_t\}_{t=0}^{s} & \text{then } GT(h_s) = p_{s+1} \\ \text{If } h_s \neq \{p_t\}_{t=0}^{s} & \text{then } GT(h_s) = 0 \end{array}$$

To any target path $\{(p_t, \alpha_t)\}_{t\geq 0}$ corresponds a grim-trigger strategy. Using the previous notations to any path $\{(p_t, \alpha_t)\}_{t\geq 0}$ corresponds a per period profit path $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 0}$.

We call the profit path $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 0}$ sustainable if the corresponding grim-trigger strategy profile is SPE. Any deviation from the corresponding path $\{(p_t, \alpha_t)\}_{t\geq 0}$ punished by infinite Nash reversion is not profitable. Formally:

Definition $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 0}$ is sustainable if:

$$\begin{array}{ll} \text{for all } \ t \geq 0 \ : & \\ (1-\delta_A)(\pi_{A,t} + \pi_{B,t}) \leq \Pi_{A,t} & \text{S(A,t)} \\ (1-\delta_B)(\pi_{A,t} + \pi_{B,t}) \leq \Pi_{B,t} & \text{S(B,t)} \end{array}$$

S(A,t) refers to the sustainability condition of firm A at time t. Firm A's continuation profit should be higher than the best deviation profit at time t which is (essentially) $(\pi_{A,t} + \pi_{B,t})$, the period t market price.

By definition if $(\Pi_{A,0}, \Pi_{B,0})$ can be obtained via a sustainable path $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 0}$, then $(\Pi_{A,0}, \Pi_{B,0})$ is a SPE payoff. In the next section we analyze the best $(\Pi_{A,0}, \Pi_{B,0})$ which can be constructed from a sustainable path.

2.3 Strategies under incomplete information

In the incomplete information game we analyze equilibria from the separation period. This restriction is motivated in section 4, where we first show that pooling equilibria yield low payoffs, and that equilibria where separation is late (in mix or pure strategy) mechanically imply a gradual increase in prices.

The objective of the analysis is to show that the transition phase can be modeled as a distortion linked with the informational problem. Formally, the goal of the analysis form section 4, is to characterize the Pareto frontier of payoffs achievable in equilibrium from the separation periods. And show (in section 5) that the best payoffs are constructed from equilibrium path featuring gradual increase in prices.
In section 6 we will describe in more details the properties of this transition phase.

The equilibrium concept used is the Perfect Bayesian equilibrium (PBE). A profile of strategies constitutes a PBE if strategies are sequentially rational (each firm plays optimally at each information sets given their information about their rival's type), and if firm's beliefs are compatible : obtained from Bayes rule whenever possible. In information sets where Bayes rule doesn't apply (referred as "off-path" or "off-schedule" information sets) the beliefs are set by the analyst.
In this model the PBE concept needs no refinement. The class of equilibra considered punishes any off-path or off-schedule deviations with infinite reversion to Nash play. This punishment scheme delivers the minmax payoff to each firm (whatever is their type), and is also a PBE for any beliefs (Nash equilibrium conditions are unaffected by the discount rate). Therefore, setting beliefs on information sets reached with 0 probability is not an issue.

A formal presentation of the class of strategies studied is presented in section 4. The next section presents the analysis of the game under complete information.

3 Pareto frontier under complete information

In this section we characterize the optimal equilibrium payoffs for the game under complete information. This game has already been analyzed in Harrington (1988) for stationary strategies and more recently for general strategies by Obara and Zincenko (2017). Thus, we proceed heuristically, recalling the key results and applying them to our duopoly case.

The first subsection treats of a necessary condition to sustain supra-competitive prices, for a given pair of discount rate. Although already proved in Obara and Zincenko (2017)[7], the technique used is different.

The second subsection presents the characterization of the frontier for firm A and B(δ_H).

3.1 Necessary condition for sustainability

Assume for this subsection only that (δ_A, δ_B) is the pair of discount rate for firm A and B. In the standard symmetric model, supra-competitive outcomes are sustainable only if the discount rate is above $\frac{1}{2}$. We here generalize this result allowing asymmetry in the discount rate.

We recall the definition of a sustainable path:

Definition A path $\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}$ is sustainable if

$$
\begin{aligned}
&\text{for all } \ t \geq 0 \ : \\
&(1-\delta_A)(\pi_{A,t}+\pi_{B,t}) \leq \Pi_{A,t} && \text{S(A,t)} \\
&(1-\delta_B)(\pi_{A,t}+\pi_{B,t}) \leq \Pi_{B,t} && \text{S(B,t)}
\end{aligned}
$$

Remark

By construction: $\Pi_{A,t} = (1-\delta_A)\pi_{A,t} + \delta_A \Pi_{A,t+1}$. Sustainability conditions can be rewritten:

$$
\begin{aligned}
\Pi_{B,t} &\leq \delta_B \Pi_{B,t+1} + \frac{1-\delta_A}{1-\delta_B}\delta_A \Pi_{A,t+1} && S(A,t) \\
\Pi_{A,t} &\leq \delta_A \Pi_{A,t+1} + \frac{1-\delta_B}{1-\delta_A}\delta_B \Pi_{B,t+1} && S(B,t)
\end{aligned}
$$

Or equivalently:

$$
\begin{pmatrix} \Pi_{A,t} \\ \Pi_{B,t} \end{pmatrix} \leq \begin{pmatrix} \delta_A & \frac{1-\delta_B}{1-\delta_A}\delta_B \\ \frac{1-\delta_A}{1-\delta_B}\delta_A & \delta_B \end{pmatrix} \begin{pmatrix} \Pi_{A,t+1} \\ \Pi_{B,t+1} \end{pmatrix} \qquad S(t)
$$

Where $X \leq Y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Because $S(t)$ is true for all t this implies[8] that for any t:

$$
\begin{aligned}
&\begin{pmatrix} \Pi_{A,0} \\ \Pi_{B,0} \end{pmatrix} \leq \begin{pmatrix} \delta_A & \frac{1-\delta_B}{1-\delta_A}\delta_B \\ \frac{1-\delta_A}{1-\delta_B}\delta_A & \delta_B \end{pmatrix}^t \begin{pmatrix} \Pi_{A,t} \\ \Pi_{B,t} \end{pmatrix} \\
\iff &\begin{pmatrix} \Pi_{A,0} \\ \Pi_{B,0} \end{pmatrix} \leq P \begin{pmatrix} 0 & 0 \\ 0 & (\delta_A+\delta_B)^t \end{pmatrix} P^{-1} \begin{pmatrix} \Pi_{A,t} \\ \Pi_{B,t} \end{pmatrix}
\end{aligned}
$$

Where :

$$
P = \begin{pmatrix} \delta_A(1-\delta_B) & 1-\delta_B \\ -\delta_B(1-\delta_A) & 1-\delta_A \end{pmatrix}
$$

[7] See theorem 3.1 working paper version

[8] The relation $\leq$ defined above is an incomplete order relation. The implication is true using the transitivity property of the relation.

If $\delta_A + \delta_B < 1$, and because payoffs are bounded, it implies that the only payoff satisfying sustainability conditions is the Nash equilibrium payoff[9]. Therefore, the only SPE is the repeated play of Nash price. Equivalently, collusion is not sustainable when $\delta_A + \delta_B < 1$.

Hence a necessary condition for sustainability of collusion is $\delta_A + \delta_B \geq 1$. This condition is also sufficient[10].

For the following parts, we will assume that $\delta_H + \delta_L < 1$. Hence there is no possible collusion when firm B is impatient. Thus, once firm A learns that firm B discount future at rate δ_L then the repeated play of the Nash equilibrium is the only continuation equilibrium. Moreover we assume $\delta_H \geq \frac{1}{2}$, so collusion is sustainable for firm A and firm B(δ_H). We now characterize the production frontier when discount factors are homogeneous (and $\delta \geq \frac{1}{2}$).

3.2 Pareto Frontier

Let $\delta_H \geq \frac{1}{2}$. We formally define the set of equilibrium payoff under complete information for the high type firms, and call F^* its frontier.

Definition *SPE set of payoffs*
Let Σ be the set of payoffs achievable in equilibrium:

$$\Sigma = \{(\Pi_{A,t}, \Pi_{B,t}) \ \text{ s.t S(A,s) and S(B,s) holds for all } \ s \geq t\}$$

And denote by F^* the Pareto frontier of this set: $F^* = F\{\Sigma\}$.

The frontier problem is stationary as the set of equilibrium payoff is unchanged given the starting date considered. To characterize the frontier we first show that equilibrium payoff where the industry profit is not r cannot be on the frontier. We also show that if the industry profit is not r, there is a local Pareto improving variation of the collusive path. We use the uniform norm as the notion of distance : $\|\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}\|_\infty = \sup_{t\geq 0}\{\|(\pi_{A,t}, \pi_{B,t})\|\}$.

Remark

Because $\pi_{k,t} \geq 0$ and $\pi_{A,t} + \pi_{B,t} \leq r$ we have $\|(\pi_{A,t}, \pi_{B,t})\| \leq r$ and thus $\|\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}\|_\infty \leq r$.

Proposition 3.1 *Assume* $\Pi_{A,0}+\Pi_{B,0} < r$ *for a path* $\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}$. *There exists a path* $\{\pi'_{A,t}, \pi'_{B,t}\}_{t\geq 0}$ *arbitrarily close to* $\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}$ *which yields Pareto improved payoffs.*

Proof See appendix A

Any equilibrium payoff which does not sum to the monopoly price cannot be on the frontier. This is not the simplest way to characterize the frontier but this result is helpful for the next sections. Eventually we show which equilibrium payoffs with r as industry profit are on the frontier.

From $\Pi_{A,0} + \Pi_{B,0} = r$, we have by construction $\Pi_{A,t} + \Pi_{B,t} = r$ for all $t \geq 0$. Sustainability conditions are equivalent to:

$$\begin{aligned} &\forall \ t \geq 0 \ : \\ &\Pi_{B,t} \leq \delta_H r \\ &\Pi_{A,t} \leq \delta_H r \end{aligned}$$

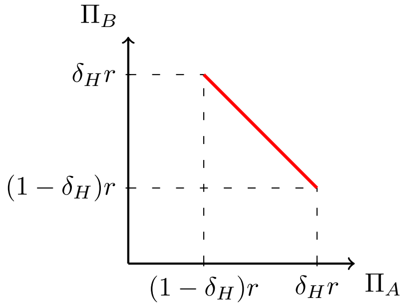
[9] The claim can be rewritten using the concepts defined in Abreu, Pearce and Stachetti (1990). Definition of the operator $B()$ as well as theorem 1 (self-generation) can be directly extended for heterogeneous discounting. Given these concepts, this claim can be restated as: if $\delta_A + \delta_B < 1$ the only bounded and self-generating payoff set is $\{(0,0)\}$.

[10] As shown in Harrington (1988) or in Obara and Zincenko (2017). In fact Grim-trigger strategies at constant price can be shown to form equilibrium profile of strategies for well-chosen constant market shares.

All the payoffs such that: $\Pi_{A,0} + \Pi_{B,0} = r$, $(1-\delta_H)r \leq \Pi_{B,0} \leq \delta_H r$ are sustainable (using a constant path $\pi_{A,t} = \Pi_{A,0}$ for example). And those points cannot be Pareto dominated, as the industry profit cannot be higher than r.

Therefore the Pareto frontier is:

$$F^* = \{(r - \Pi_{B,0}, \Pi_{B,0}) \ : \ (1-\delta_H)r \leq \Pi_{B,0} \leq \delta_H r\}$$



4 Equilibria under incomplete information

4.1 Parameter space

We assumed that $\delta_L + \delta_H < 1$ and $\delta_H \geq \frac{1}{2}$. So that if firm B's type is δ_L the only equilibrium is the repeated play of the Nash equilibrium. And if firm B's type is δ_H then any price can be sustained. Given this choice a pooling equilibrium can only be repeated play of the Nash equilibrium. Hence, separation is required for high type firms to make strictly positive profit. Ex-post, we find out our analysis holds even if the high type firm B doesn't have the same discount rate of firm A, but still the sum is above one.

Another choice could be $\delta_L + \delta_H \geq 1$, $\delta_L < \frac{1}{2} < \delta_H$. That is collusion is always sustainable, but firm δ_L needs a market share of at least $1 - \delta_L > \frac{1}{2}$ for prices to be sustainable. Hence firm A is still willing to know B's type as she can increase its market share in equilibrium. However firm δ_H is always better off claiming to be of type δ_L as its market share increases. Thus firm δ_H is never willing to reveal its type.

4.2 Strategies

Given the parameter space pooling equilibria are unattractive. Indeed, the only price path who is sustainable for firm A and $B(\delta_L)$ at the same time is the repeated play of the Nash price, hence from section 3 and given the parameter space, pooling equilibria yield minmax payoffs for all three firms.[11].

We focus on equilibria in pure strategies, where separation occurs in period 0. This is an important restriction. However we claim that this class of equilibria is the least likely to produce a "transition phase".

Late revealing implies that the impatient firm is willing to continue, so that the equilibrium path is sustainable for some periods. However, from section 3, because $\delta_H + \delta_L < 1$ it means that the continuation value of both firms has to be strictly increasing, thus the price path must be strictly increasing

[11] remark that when types are revealed, if B is impatient the only continuation equilibria is the repeated play of the Nash price. Hence, equilibrium conditions for a pooling price path are the same as a sustainability conditions for δ_L firm. Also, the best way to enforce a path for firm A is still reversion to Nash price punishment, and so equilibrium conditions from a pooling price path are also the same as sustainability conditions for firm A (lack of information plays no-role as firm B and $B(\delta_L)$ plays the same prices). Thus from section 3 the pooling path can only be the repeated play of the Nash price.

as well. Therefore, late revealing (either in mix or pure strategy) implies that the equilibrium path exhibits a "transition phase", where prices gradually increase.

Our objective is to characterize the transition phase as a distortion resulting from patient firms screening each other's type. Our analysis shows that the gradual increase of prices is an efficient way to deter impatient firms from mimicking, while ensuring Pareto optimal profits for patient firms.

Furthermore, we consider strategies where out schedule and on schedule deviations are punished by infinite Nash reversion. This is not a restriction as it is the worst punishment. In addition, this punishment is BPE for any belief on gives to firm A, hence we do not specify firm's A belief out of path.

Given this restrictions, we can prove the following lemma that restrict the structure of equilibrium candidates:

Lemma 4.1 *In an equilibrium where types are separated in period* 0*:*

i) When prices are positive, there is no loss in considering collusive path where firm A and B set the same price in each period.

And paths that achieve Pareto payoff are such that:

ii) the price set by firm A is non-negative, at any period.
iii) the price set by firm B is non-negative at any period after separation.

Proof *See appendix B*

From period 1 on, we can restrict our attention to positive prices. Thus we use the same notations as in section 3. That is $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 1}$ refers to the per-period profits induced by the collusive path from period 1, where $\pi_{A,t} + \pi_{B,t} \geq 0$ is the market price and so the static best response payoffIn the next section, we formally define the class of strategies studied and conditions for these to form a PBE profile.

4.3 Equilibrium conditions

We focus on strategies where firm $B(\delta_L)$ reveals its type in period 0. From period 1 on, if both firms are patient the strategies are standard grim-trigger strategies with positive prices[12](and so we adopt the same notations as defined in section 2).

Definition *Separating Strategy*
Given a collusive path $\{(\pi_{A,t}, \pi_{B,t})\}_{t\geq 1}$ we consider the following strategies:

- If $h_t = \{(p_{A,0}, p_{B,0})\,\{(\pi_{A,s}, \pi_{B,s})\}_{s=1}^{t}\}$ firm A and B(δ_H) play $(\pi_{A,t+1}, \pi_{B,t+1})$). Otherwise firm A and B play 0.

- Firm δ_L slightly undercut $p_{A,0}$ (plays the static best response, $p_{A,0} \geq 0$ from previous lemma). If $h_t \neq \{(\pi_{A,s}, \pi_{B,s})\}_{s=0}^{t}$ it plays the Nash price.

- On $h_t = \{(p_{A,0}, p_{B,0})\,\{(\pi_{A,s}, \pi_{B,s})\}_{s=1}^{t}\}$, Firm δ_L plays the optimal mimicking strategy.

In period 0, the beliefs of firm A is ρ. From period 1 $\rho_t \in \{0, 1\}$, because action of $B(\delta_H)$ and $B(\delta_L)$ differs in period 0. On the decision node reached with probability 0, the Nash price is played and so beliefs do not affect equilibrium conditions. The set of compatible beliefs do not change in the following sections.

[12]This is without loss from Lemma 4.1.

Remarks

Given the definition of the strategy we can write the associated payoffs for each firm. Firm B's payoff is simply $\Pi_{B,0} = (1-\delta_H)(1-\alpha_0)p_{B,0} + \delta_H \Pi_{B,1}$. Firm A's payoff is $\Pi_{A,0} = (1-\delta_H)\alpha_0 p_{A,0} + \delta_H \Pi_{A,1}$ in case firm B is of type δ_H, which for firm A occurs with probability ρ. And if firm B is δ_L A receives 0 in equilibrium. Hence we have $\mathbb{E}[\Pi_A] = \rho \Pi_{A,0}$. Firm δ_L receives $p_{A,0}$ in equilibrium. Also we assumed firm δ_L is playing optimally when she is mimicking. Hence for a given history $h_t = \{(p_{A,0}, p_{B,0}) \{(\pi_{A,s}, \pi_{B,s})\}_{s=1}^{t}\}$, δ_L firm δ_L's payoff can be written as: $(1-\alpha_0)p_{B,0} + \max_{k\geq 1}\left\{\sum_{t=1}^{k} \delta_L^t \pi_{B,t} + \delta_L^k \pi_{A,k}\right\}$.

After separation, equilibrium conditions are standard sustainability conditions (see section 3). We focus on the equilibrium conditions in period 0 :

Definition *Incentive compatibility*

$$\begin{aligned}
&\text{S(B,0)} : \Pi_{B,0} \geq (1-\delta_H)p_{A,0} \\
&\text{IC}_L : \alpha_0 p_{B,0} + \max_{k\geq 1}\left\{\sum_{t=1}^{k} \delta_L^t \pi_{B,t} + \delta_L^k \pi_{A,k}\right\} \leq p_{A,0} \\
&\text{IC}_A : \rho \Pi_{A,0} \geq (1-\delta_H)\max\{p_{B,0}, (1-\rho)p_{A,0}\}
\end{aligned}$$

Because firm $B(\delta_L)$ is required to play her static best response, only the "on-schedule" deviation (mimicking firm $B(\delta_H)$) is a relevant condition. This condition is labeled IC_L. By the same token, as the discount rate does not affect the static best response of a firm, firm B(δ_H) period 0 equilibrium condition on-schedule (mimicking firm $B(\delta_L)$) and off-schedule (static best response) are equivalent. Hence firm B period 0 condition is a standard sustainability condition.

Firm A's static best deviation in period 0 varies whether $p_{B,0} < 0$ (and so strictly lower than $p_{A,0}$, or $p_{B,0} \geq 0$ ($= p_{A,0}$ wlog). Firm's A best deviation is to undercut slightly firm $B(\delta_L)$ in order to : in the first case obtain essentially $p_{A,0}$ profit with probability $1-\rho$, and in the second case obtain $p_{A,0} = p_{B,0}$ with probability 1. Hence the payoff of firm's A best deviation in 0 can be written $\max\{p_{B,0}, (1-\rho)p_{A,0}\}$, which in equilibrium must be lower than the expected profit of firm A $\rho\Pi_{A,0}$.

To present the main intuition it is relevant to subjectively distinguish two subclasses of equilibria.

When $p_{B,0} < 0$, the way to sort types relies on a "money-burning" idea. Firm $B(\delta_H)$ undergoes losses in the first period which firm $B(\delta_L)$ cannot profitably replicate. The impatient firm weights more the present : for a sufficiently low $p_{B,0}$ she is not willing to mimic.

When $p_{B,0} > 0$, firm $B(\delta_L)$ is induced to reveal her type as she can claim a "rent" she is expected to undercut on path the market price. This is more similar to a "screening" way to sort type, as the type that is willing to mimic the other receives a rent.

We will analyze the "money-burning" strategy in section 6. We explain with simple examples the tension and trade offs which are inherent to the "screening" way to sort types.

Examples

First, we try a price path which play the monopoly price each period. We know from the first part that any best payoff can be implement with as constant path price where $(1-\delta_H)r \leq \pi_{B,t} \leq \delta_H r$ and $\pi_{A,t} + \pi_{B,t} = r$. In that case IC_L is always satisfied[13]. But IC_A may fail to hold. Because $\Pi_{A,0} \leq \delta_H r$ a necessary condition for IC_A to be satisfied is $\rho \geq \frac{1-\delta_H}{\delta_H}$.

[13]To be willing to continue mimicking on a constant path, firm $B(\delta_L)$ must claim at least a share $1-\delta_L$ of the per period profit. Because $\delta_L + \delta_H < 1$ and the highest share $B(\delta_H)$ can claim is δ_H, undercutting as early as possible is optimal.

Intuitively if ρ is too low then firm A does not expect collusion to occur and may prefer to pocket itself the period 0 rent by deviating. Hence the first tension comes from the fact that IC_A is limiting the rent that δ_L can get, because firm A may be willing to get it itself if collusion is too unlikely.

A second try is to set the highest possible rent and then play the first best path, (so that IC_A is met). Intuitively, firm δ_L may then be willing to mimic. Set the highest possible rent, we can assume firm A gets the highest share on the collusive path, we have
$(1-\delta_H)(\pi_{A,t}+\pi_{B,t}) = \rho\delta_H r$. Plug in IC_L and we argue for a deviation in period 1:
$\rho\frac{\delta_H}{1-\delta_H}r \geq \pi_{B,0} + \delta_L r$. If ρ is too low then $B(\delta_L)$ should mimic.

To sum up: first the rent is limited once ρ is low, second once the rent is low firm $B(\delta_L)$ is willing to mimic. Hence one has to reduce the next prices to reduce the mimicking incentives, but this bounce back in IC_A and the rent has to be reduced again. This tension is peculiar compared to classical mechanism problem. Indeed, to relax incentive compatibility conditions usually the mechanism increases the difference in the allocation between high types and low types, this is based on strict increasing differences property of the value functions. However in our case it appears that reducing future prices relax the incentive compatibility constraints between low types and high types. The following discussion explicit this intuition.

Consider a variation of the price at time t, from date 0 standpoint the payoffs caries as such:

$$\begin{aligned}
&\frac{\rho d\Pi_{A,0}}{1-\delta_H} = \rho\delta_H^t d\pi_{A,t} \\
&d\Pi_{L,0} = 0 \;\; \text{if the max is achieved strictly before t} \\
&d\Pi_{L,0} = \delta_L^t(d\pi_{B,t} + d\pi_{A,t}) \;\; \text{if the max is achieved at t} \\
&d\Pi_{L,0} = \delta_L^t d\pi_{B,t} \;\; \text{if the max is achieved strictly after t}
\end{aligned}$$

Hence for small ρ if the maximum is achieved at t reducing the price of period t may relax the incentive compatibility constraints. Moreover those variations depend on t and for all ρ there is a t sufficiently large for which we have strict increasing differences (as δ_L^t is negligible with respect to δ_H^t).

To sum up, the path variations which relaxes IC's may be negative or positive depending on t and ρ. Furthermore, for t large enough we always have strict increasing differences for any level parameters. Hence one can preview the following distortions of a path that meets the IC's: first period prices are reduced, while for some large t the path should set the highest possible prices. Therefore, to met the IC's, the first best collusion level has to be delayed.

Those variations apply also to the "money-burning" strategy. However, it is not clear why firm may be willing to delay for more than one period. It is possible to achieve first best price right from period 1 on, if $p_{B,0}$ is negative enough. We explain in more details why it is optimal to have more delay in section 6.

Nevertheless, in both strategies, the patient firm path from period 1 on must maximize their payoffs while restricting as much as possible the mimicking profit of δ_L, in order to ease separation. We define this problem as the "optimal use of instruments". We treat this problem in the following section. We will show that the solution feature a transition phase, to which we give its optimal speed and length.

Before analyzing the Pareto frontier of these strategies, we first analyze the Pareto frontier of the optimal use of instrument problem, which is common to any of these strategies.

5 Optimal use of Instruments

To ease separation, a path designed for patient firms gives the best Pareto payoff to the high type firms, and restricts the payoff a low type firm can have. This defines the optimal use of instrument problem. Following the mechanism design vocabulary, instruments refers to additional degree of freedom that are used in order to increase the difference in profits between types, hence facilitating separation of types.

Formally, the Optimal use of instrument problem is defined after the separation period, as the analysis of Pareto optimal sustainable paths for patient firms, such that a low type firm payoff on those paths cannot excess $\Pi_{L,1}$.

Definition *Optimal use of Instruments*

$\mathcal{C}(\Pi_{L,1}) \subset \mathbb{R}^2$ is the set of patient firm's payoffs obtained by sustainable paths yielding less than $\Pi_{L,1}$ profit for an impatient firm:

$(\Pi_{A,1}, \Pi_{B,1}) \in \mathcal{C}(\Pi_{L,1})$ if there exist a path $\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}$ yielding $(\Pi_{A,1}, \Pi_{B,1})$ s.t $\forall t \geq 1$,

$$(1-\delta_H)(\pi_{A,t}+\pi_{B,t}) \leq \Pi_{A,t} \qquad \text{S(A,t)}$$
$$(1-\delta_H)(\pi_{A,t}+\pi_{B,t}) \leq \Pi_{B,t} \qquad \text{S(B,t)}$$
$$\Pi_{L,1} \geq \max_{k\geq 1} \left\{ \sum_{t=1}^{k} \delta_L^{t-1}\pi_{B,t} + \delta_L^{k-1}\pi_{A,k} \right\} \qquad \underline{L}$$

The Pareto optimal use of instrument problem is defined as the analysis of the Pareto frontier of $\mathcal{C}(\Pi_{L,1})$:

$$F(\Pi_{L,1}) \overset{def}{=} F\{\mathcal{C}(\Pi_{L,1})\}$$

Notice that without the last constraint $\underline{L}$, this problem is equivalent to a perfect information problem as in section 3. The last constraint affects all future periods. We simplify the problem by decomposing $\underline{L}$ into per-period constraints, and by introducing a state variable.

Proposition 5.1 State Variables
A sustainable path $\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}$ yields payoff for an impatient firms lower than $\Pi_{L,1}$ if and only if there exist a sequence of state variables $\{\Pi_{L,t}\}_{t\geq 1}$ such that, for all $t \geq 1$:

$$i)\Pi_{L,t+1} = \frac{\Pi_{L,t} - \pi_{B,t}}{\delta_L} \qquad \textit{Law of Motion}$$
$$ii)\Pi_{L,t} \geq \pi_{A,t} + \pi_{B,t} \qquad \underline{L}(t) \quad \textit{Non-mimicking conditions}$$

Proof See appendix C

We pause here to explicit the method. The state $\Pi_{L,t}$ is an upper-bound on $B(\delta_L)$'s profit from period t on. Hence it has to satisfy two conditions, a) it must be above the undercutting profit of period t ; b) if firm $B(\delta_L)$ mimics one more period it connects to the upper-bound at $t+1$: $\Pi_{L,t} = \pi_{B,t} + \delta_L \Pi_{L,t+1}$. The first condition gives the Non-mimicking conditions. The second condition tie the future state with the choice of today's profit for firm B: "Law of motion". When the profit given to firm $B(\delta_H)$ increases in t, the mimicking payoff of $B(\delta_L)$ increases, as a result the future upper-bound has to be lowered.

This method simplifies the analysis. First, the constraint $\underline{L}$ encapsulates a complicated maximization problem with as many degree of freedom as there are periods and patient firms. It is replaced by per-period constraints, with only two degrees of freedom. Second, it allows us to set up the problem in a recursive form.

To do so, we introduce additional notations.

Definition *Current Constraints Set*
$(\pi_{A,t}, \pi_{B,t}) \in \mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right)$ if:

$$\begin{aligned} (1-\delta_H)\pi_{B,t} &\leq \delta_H \Pi_{A,t+1} && \text{S(A,t)} \\ (1-\delta_H)\pi_{A,t} &\leq \delta_H \Pi_{B,t+1} && \text{S(B,t)} \\ \Pi_{L,t} &\geq \pi_{B,t} + \pi_{A,t} && \underline{L}(t) \end{aligned}$$

To formalize the recursive nature of the problem, we exhibit a principle of optimality. Given $\Pi_{L,1}$, solving for a path yielding Pareto payoff to patient firms is equivalent to: first pick Pareto optimal tomorrow's continuation values given a state $\Pi_{L,2}$, second choose an optimal today's flow profits taking into account how $\Pi_{L,2}$ is affected (and thus how tomorrow's optimal continuation values are affected). This idea is presented in the following proposition.

Proposition 5.2 Recursivity

For all $\Pi_{L,t} \geq 0$ *we have:*

$$i) F(\Pi_{L,t}) \neq \emptyset$$

$$ii) F(\Pi_{L,t}) = F\left\{(1-\delta_H)\mathcal{I}\left(\Pi_{L,t}, .\right) \oplus \delta_H F(\Pi_{L,t+1}(.))\right\}$$

Where $\mathcal{I}(\Pi_{L,t}, .)$ and $F(.)$ are the following correspondences:

$$\begin{aligned} \mathcal{I}(\Pi_{L,t}, .) \colon \mathbb{R}^2 &\rightrightarrows \mathbb{R}^2 \\ (\Pi_{A,t}, \Pi_{B,t}) &\mapsto \mathcal{I}(\Pi_{L,t}, (\Pi_{A,t}, \Pi_{B,t})) \end{aligned}$$

$$\begin{aligned} F(.) \colon \mathbb{R}^2 &\rightrightarrows \mathbb{R}^2 \\ (\pi_{A,t}, \pi_{B,t}) &\mapsto F\left(\frac{\Pi_{L,t} - \pi_{B,t}}{\delta_L}\right) \end{aligned}$$

$\oplus$ is an operator defined before, and $\delta_H F$ refers to a standard scalar multiplication of a set.

Proof See appendix C

Point $ii)$ corresponds to the principle of optimality: in order to be Pareto optimal, a path has to pick tomorrow's continuation value on the frontier.With the Law of motion, it also exhibits an important trade-off. Reducing firm B profit's today increases the future states (as it reduces the mimicking profit). By increasing the future states, tomorrow's continuation values may be pick in a Pareto higher frontier. Therefore, Pareto optimal path balance the current cost for firm B to deter firm δ_L from mimicking, with the future gain of picking higher continuation values.

The future benefit of reducing $\pi_{B,t}$ depends on the number of non-mimicking conditions which are binding: the higher is the number of constraints that are effectively relaxed, the higher is the benefit.

The analysis focuses on solving this trade off, especially, what is the minimal number of constraints binding for which reducing $\pi_{B,t}$ is Pareto improving.

To do so we start with a series of lemmas which specify the structure of paths that yields Pareto optimal payoffs: "sorting conditions". Using this, we are able to solve the trade off and characterize the frontier.

5.1 Preliminaries

In this section we present results which show that any path yielding an optimal payoff has a specific structure. The sorting condition lemmas explicit this structure.

The following lemma shapes the path yielding optimal payoffs. At an optimal path, if some mimicking conditions are binding then it must be the first one that binds. That is firm $B(\delta_L)$ cannot strictly prefer deviating in $t+1$ rather than in t. If that is true then we can increase the profit of firm A in period t without violating constrains. In other words, if $B(\delta_L)$ strictly prefer to mimic, it is always possible to fill this gap with firm's A profit.

Lemma 5.3 (Sorting condition for $\underline{L}$) $(\Pi_{A,t}, \Pi_{B,t}) \in F(\Pi_{L,t})$ *are obtained from paths such that:*

i) $\underline{L}$*(t+s) are binding for all* $s \geq 0$ *if and only if* $\Pi_{L,t} = 0$,

ii)$\underline{L}$*(t+s) are slack for all* $s \geq 0$ *if and only if* $\Pi_{L,t} > r$

iii) $\exists\ l \geq 1$ *s.t* $\underline{L}(t+s)$ *is binding* $\forall\ 0 \leq s < l$ *and* $\underline{L}(t+s)$ *are slack* $\forall\ s \geq l$ *if and only if* $0 < \Pi_{L,t} \leq r$,

Proof See appendix D

From this lemma we know that any optimal payoffs can be obtained with a paths such that only the first $\underline{L}$ are binding. Moreover when $\underline{L}(.)$ binds the price is lower than r. Hence the number of non-mimicking conditions correspond to the number of periods required for the path to reach r. Thus, we will call l the number of non-mimicking condition the "path length". That is the number of periods required to reach r, after which the problem is equivalent to the complete information frontier problem.

Definition *Transition phase Length*
Consider an optimal path $\{(\pi_{A,s}, \pi_{B,s})\}_{s\geq t}$ for a given $\Pi_{L,t}$. The transition phase length noted l is the number of binding $\underline{L}$. Consequently, in the first l periods the price is below r, and is equal to r after l periods.

Since only the first non-mimicking conditions bind, we can define the speed at which the price path increase. If the price is below r the price path has to bind the non-mimicking condition, an thus increases at the same rate the mimicking conditions are relaxed. δ_L is the maximal rate: firm's B is not receiving any profit and this the mimicking gains are minimum. $\delta_L + \delta_H$ is the lowest speed for optimality reasons: if the speed is too low i.e $\pi_{B,t}$ is high one can find a reallocation of profits which reduces $\pi_{B,t}$ and increases future profits from B so that both profits increases due to non-mimicking being relaxed.

Lemma 5.4 (Speed of price increase) *For an optimal path, during the transition phase, the speed of price increase is bounded by two geometric speed. Formally, for* $t \leq l-2$ *we have:*

$$\frac{1}{\delta_H + \delta_L} \leq \frac{\pi_{A,t+1} + \pi_{B,t+1}}{\pi_{A,t} + \pi_{B,t}} \leq \frac{1}{\delta_L}$$

Proof See appendix D

To sum up this part, we have showed that any optimal payoff can be achieved by a path where the first l non-mimicking conditions bind. Hence l is the time required for the path to achieve r the monopoly price. One can see l as a rough measure of distortions. As the rent gets lower l increases: the distortion is higher. In the characterization section we will associate to any optimal path the number l of non-mimicking conditions.

Intuitively, the higher is l the higher is the benefit of reducing the present $\pi_{B,t}$. We present additional structure to characterize this trade off. For firm A there is no tension: reducing $\pi_{B,t}$ is not costly while the future benefits are positive. Hence to characterize this trade off we need to understand what are the paths that favor firm $B(\delta_H)$. Thus we turn to the analysis of $S(A,.)$ conditions and explicit sorting conditions.

Lemma 5.5 Sorting conditions for S(A,.)
Any $(\Pi_{A,t}, \Pi_{B,t}) \in F(\Pi_{L,t})$ *are obtained from paths such that:*

For any length we have:
i)If S(A,t+s) is binding for a $s \geq 0$ *then S(A,t+s+1) is binding*

And for any length greater than 2 we have :
ii)If S(A,t+s) is binding for a $0 \leq s \leq l-2$ *then*

$$\pi_{A,t+s} = \frac{\delta_L \Pi_{L,t+s}}{\delta_L + \delta_H}, \quad \pi_{B,t+s} = \frac{\delta_H \Pi_{L,t+s}}{\delta_L + \delta_H}, \quad \Pi_{L,t+s+1} = \frac{\Pi_{L,t+s}}{\delta_L + \delta_H}$$

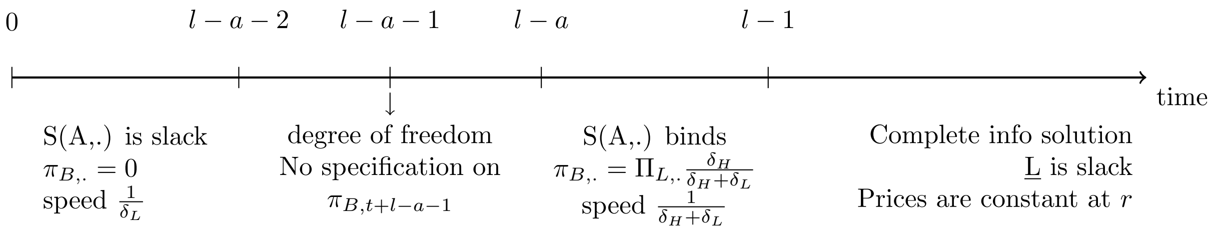
iii)If S(A,t+s) is slack for a $1 \leq s \leq l-1$ *then*

$$\pi_{A,t+s-1} = \Pi_{L,t+s-1}, \quad \pi_{B,t+s-1} = 0, \quad \Pi_{L,t+s} = \frac{\Pi_{L,t+s-1}}{\delta_L}$$

Proof See appendix D

In the first point, the path is just identified because we have enough condition binding. For point ii) : if a sustainability condition is slacking for A an t, then $\pi_{B,t-1} = 0$. This is similar to an optimality principle. If one wants to maximize firms B profit at t then it is better to take the best profit of B at $t+1$ before setting the optimal $\pi_{B,t}$. Hence if S(A,t+1) is slack this means that we are not taking the best profit for firm B and thus $\pi_{B,t} = 0$ to reduce the inefficiencies.

From the two sorting conditions, a path which yields Pareto payoff can be indexed by its transition length l, and the number of S(A,.) conditions binding (noted a) during the transition phase. All the path with the same (l, a) index have only one degree of freedom (in the choice of market shares in the period $l-a-1$). The following diagram sums up the two previous lemma:



Consequently, all the payoffs obtained from a path indexed by (l, a) and at a given state $\Pi_{L,t}$ are located on the same segment. This motivates the following definition.

Definition *Partition of the Frontier*
$F_{a,l}(\Pi_{L,t}) \subset F(\Pi_{L,t})$ is the set of payoffs which can be obtained from paths of transition length l and with $a \leq l$ S(A,.) constraints binding.

In the following section we characterize the frontier.

5.2 Characterization

The goal is now to determine which paths indexed by (l, a) yield frontier payoff ; and which index (l, a) is Pareto optimal. The argument is conducted using the recursivity property of the problem, which is stated in assertion 1. The Pareto optimality of paths is assessed by induction starting from $F_{0,0}(\Pi_{L,t}) = F^* \iff \Pi_{L,t} > r$. Indeed, the $(0,0)$ segment is trivially optimal (provided the state is sufficiently high so that this segment exists) as it is the Pareto frontier of section 3.

Paths indexed by (l, a) may not always exist at a given $\Pi_{L,t}$. Indeed l implies the market price to be below r for l period, and a defines the increase rate, thus the state $\Pi_{L,t}$ must be precisely set ,for the price path to exist. Computation on conditions on $\Pi_{L,t}$ for the segment (l, a) to exist are presented in the appendix. There are also necessary conditions for $F_{l,a}$ to be non-empty. One can show:

$i)$For $l = a$

$F_{(l,l)}(\Pi_{L,t}) \neq \emptyset$ only if $(\delta_L + \delta_H)^l r < \Pi_{L,t} \leq (\delta_L + \delta_H)^{l-1} r$

$ii)$For $l > a > 0$:

$F_{(l,a)}(\Pi_{L,t}) \neq \emptyset$ only if $\delta_L^{l-a}(\delta_H + \delta_L)^a r < \Pi_{L,t} < \delta_L^{l-a-1}(\delta_L + \delta_H)^a r$

$iii)$For $l > a = 0$:

$F_{(l,0)}(\Pi_{L,t}) \neq \emptyset$ only if $\delta_L^l < \Pi_{L,t} \leq \delta_L^{l-1} r$

The following statements are made at a given $\Pi_{L,t}$ for which the segment (l, a) considered exists.

Assertion 1 : a path (l, a) yields Pareto payoff only if it picks Pareto optimal continuation values. Thus by construction, a (l, a) path is Pareto only if the $(l-1, a)$ path is if $l > a$, or the $(l-1, a-1)$ path is when $l = a$.

When $l - 1 > a$, lemma 5.5 implies that only $\pi_{B,t} = 0$ is optimal. Thus when $l - 1 > a$ the path is Pareto if and only if the path $(l-1, a)$ is Pareto.

When $a = l$, today's decision is identified by the constraints, but has not be proven to be optimal. And when $a = l - 1$, today's decision is not fixed by the lemma, the market share can be arbitrarily chosen (provided it does not change the transition length). In both cases, the current decision has no effect on current industry profit (market sharing problem), but may affect future industry profit via the state variable.
It is the main trade-off, reducing firm B's market share is costly for firm B today but reduces the incentives of firm δ_L to mimic and so increases future profits.

Since the cost of deterring firm $B(\delta_L)$ from mimicking is completely borne by firm B, the decision of reducing firm's B market share to 0 is always Pareto, (it strictly increases firm A's profit). The decision of not reducing B's market share is Pareto, only if current cost overweight the future gains. In other words, if the segment where $(l, l-1)$ paths are located is on the frontier then it has a strictly decreasing slope (profit of firm B strictly increases as the profit of firm A decreases strictly).

Intuitively, when l is higher, then the future benefits of increasing $\Pi_{L,t+1}$ is higher, while the cost is constant. Thus, the slope of segments where $(l, l-1)$ paths are located is increasing in l. Therefore, there must be a $\overline{a}$ such that $\forall l > \overline{a}$ it is Pareto to set firm B's market share to 0. And $\forall l \leq \overline{a}$ any market share decision is Pareto (in particular the one which binds S(A,.)).

Observation : On a given segment, the Pareto optimality of the decision does not depend on $\Pi_{L,t}$. The trade-off only depends on l and so the slope of a segment (l, a) at $\Pi_{L,t}$ or $\Pi'_{L,t}$ is the same (if the segment exists in at both states). Thus, although this argument is conducted for a given $\Pi_{L,t}$ for a (l, a) path, it can be extended for all $\Pi_{L,t}$ where the (l, a) segment exists.

Therefore, starting from $F_{0,0} = F^*$, and by induction using assertion 1, all (l, l) segments are on the frontier if $l \leq \overline{a}$, and so also all (l, a) satisfying the condition. When $l - 1 \geq \overline{a}$, by induction again it must be that (l, a) is constructed from a Pareto $(a + 1, a)$ path and thus it is required that $a \leq \overline{a}$.

Therefore, provided a correct choice of $\Pi_{L,t}$, a (l, a) segment is on the frontier (and so equal to $F_{l,a}$) if and only if $a \leq \overline{a}$.

The slope of a $(l, l-1)$ segment is given by: (see appendix D for computations).

$$\frac{d\Pi_{B,t}}{d\Pi_{A,t}}(l, l-1) = \frac{\delta_H}{\delta_L}\left(\frac{\delta_H - \delta_L}{\delta_H} - \left(\frac{\delta_H}{\delta_L + \delta_H}\right)^{l-1}\right)$$

The slope of the segment is indeed independent of $\Pi_{L,t}$, and increasing in l.

Hence we can define $\overline{a}$ as the unique integer such that:

$$\left(\frac{\delta_H}{\delta_L + \delta_H}\right)^{\overline{a}} \leq \frac{\delta_H - \delta_L}{\delta_H} < \left(\frac{\delta_H}{\delta_L + \delta_H}\right)^{\overline{a}}$$

It follows that:

Proposition 5.6 *Assume $\Pi_{L,t}$ is such that the segment (l, a) is non-empty.*

A segment (l, a) is Pareto optimal and therefore equal to $F_{l,a}$ if and only if $a \leq \overline{a}$.

Proof In the main text, computation details presented in appendix D

It can be shown that $\overline{a}$ is higher than 2 for all choices of (δ_H, δ_L).

By construction, $\overline{a} + 1$ can be interpreted as the number of future $\underline{L}$ binding before the market share of firm B *has* to be set to 0. In other words, $\overline{a} + 1$ is the number of future $\underline{L}$ binding before the path *has* to "fully" deter δ_L from mimicking.
In this interpretation, the transition phase is not "bang-bang". That is the gradual increase of prices is not only determined by preventing low type firms from mimicking, it also ensures Pareto optimal profit for firms. Hence, $\overline{a}$ can be viewed as a solution to a "first order condition" in discrete time. And so the price increase is relatively "smooth", despite the fact that the problem is completely linear. This interpretation will be investigated in more details in the next section, where a Pareto Payoff frontier from period 0 is computed.

Comparative statics on $\overline{a}$ is also instructive: it increases when δ_H is reduced and δ_L increases.Indeed, when δ_H decreases, reducing $\pi_{B,t}$ is more costly and future benefit less appealing., hence the number of allowed S(A,.) binding increases as well. And if δ_L increases the future benefit of reducing B's period profit is reduced, as δ_L becomes more patient, reducing incentives to mimic is harder. Thus the number of allowed S(A,.) binding increases.

To conclude, the Pareto frontier of the arbitrage problem can be expressed as the reunion of $F_{l,a}$ for admissible pairs (l, a) (from proposition 5.6) and specific states $\Pi_{L,t}$ (condition i), ii) or iii) in the text above).

In the next section we compute the payoff frontier from period 0 in a particular case. We present more specific properties of the transition phase and the price path.

6 Optimal money-burning strategies

In this section, we characterize the payoff frontier in the particular case of $\rho = 0$.

In this case, only the money-burning strategy is possible, also firm A has to play 0 in period 0 as she doesn't expect collusion. Thus there is no feed back effects between the rent level, and the future continuation profits. The frontier is thus much simpler to compute compared to the case where $\rho > 0$. However, it will be shown that a "bang-bang" solution may not be optimal, despite the problem being completely linear[14].

[14]This problem belongs to the class of linear programming problems: objective is linear, and constraints are described by linear functions.

We formally restate the equilibrium conditions:

$$\text{S(B,0)} : \delta_H \Pi_{B,1} \geq -(1-\delta_H) p_{B,0}$$
$$\text{IC}_L : \max_{k\geq 1} \left\{ \sum_{t=1}^{k} \delta_L^{t-1} \pi_{B,t} + \delta_L^{k-1} \pi_{A,k} \right\} \leq \frac{-p_{B,0}}{\delta_L}$$
$$\text{IC}_A : p_{A,0} = 0$$

Firm $B(\delta_H)$ continuation payoff from period 1 on should exceed the losses in period 0. For firm δ_L, the mimicking payoff from period 1 on must be lower than the benefit of not incurring the period 0 loss by playing the negative price $p_{B,0}$. By the same arguments as highlighted in proposition 5.2, a path yielding Pareto payoff must be Pareto from period 1 on. In other words, from period 1 on we must pick a path in $F\left(\frac{-p_{B,0}}{\delta_L}\right)$. Formally this can be written as :

$$\mathcal{F}(0) = F\left\{ (1-\delta_H)\{(0, p_{B,0}) \mid IC_L,\ S(B,0) \text{ hold } \} \oplus \delta_H F\left(\frac{-p_{B,0}}{\delta_L}\right) \right\}$$

Where $\mathcal{F}(0)$ represents the payoff frontier from period 0, for $\rho = 0$.

6.1 Characterization

Only firm $B(\delta_H)$ incurs the deterrence cost : firm A plays a price of 0 in period 0. Therefore, to maximize firm's A profit, firm $B(\delta_H)$ sets a price low enough in order to play the first best collusive scheme from period 1 on. It will be the starting point of the construction of the frontier. Note that in this case their is no delay, the transition phase consists in only 1 period and is "bang-bang".

We explicit this part of the frontier. In order to achieve the first best from period 1 on, we need the state to be at least r, so that the minimal cost is $p_{B,0} = \delta_L r$. Because δ_L is indifferent between mimicking to deviate in period 1 and reveal in period 0 (both yield 0 payoff), then firm B strictly prefers not to deviate in period 0 as she is more patient (see lemma C.1 in appendix C for details). Therefore, the following set is a part of the frontier:

$$(1-\delta_H)\{(0, -\delta_L r)\} + \delta_H F^* \subset \mathcal{F}(0)$$

where the sum sign is the standard Minkowski sum.

We turn to the rest of the frontier, and solve whether firm B's profit can be raised.

When $p_{B,0}$ increases, it reduces the future state and thus future profits decrease, however firm B makes lower current loss. This is the same trade-off as in section 5. Therefore, we know that increasing $p_{B,0}$ does not increase firm B's profit if the following (l,l) paths has a length higher than $\overline{a}$ (weakly in this case as we start in period 0).

Proposition 6.1 Frontier characterization[15]
When $\rho = 0$, and separation occurs in period 0, the Pareto frontier payoffs achievable in equilibrium is:

$$\mathcal{F}^s(0) = \left\{ \Pi(p_{B,0}) | -\delta_L r \leq p_{B,0} \leq -\delta_L(\delta_L + \delta_H)^{\overline{a}} r \right\} \bigcup ((1-\delta_H)(0, -\delta_L r) + \delta_H F^*)$$

Proof See appendix E for detailed proof and close form formulas.

[15] $\overline{a}$ refers to the integer define in proposition 5.6

6.2 Discussion

The frontier point that favors firm A the most is "bang-bang" : the amount of deterrence in period 0 is sufficiently high for the path to jump directly to the first-best level from period 1 on. Because firm A doesn't pay the deterrence cost, there is no trade-off when favoring firm A and the distortion must be minimum.

In contrast, the frontier point that favors firm B the most has the highest level of distortion, i.e the longest transition length. There is a minimal length (which can be large depending on the choice of parameters) for which it would also be optimal for firm B to increase the level of deterrence. The issue for firm B is that she cannot appropriate the whole benefit by preventing firm δ_L from mimicking. Due to sustainability conditions, firm A has to receive a positive share of profit in equilibrium. As firm A free rides on B, firm B underinvests in deterrence. As a result, the industry profit is lower at the frontier point that favors firm B. Side payments would allow patient firms to be more efficient in their deterrence schedules.

In a non-linear model —like imperfect competition model, we claim that this effect is magnified. Under standard assumptions on payoffs, the difference between the collusive profit and deviation profit is a concave function[16]. Hence, when the path is closer to the highest sustainable price, an increase in the price (or a decrease in the quantity) increases the patient firms' collusive profits at a lower rate, but increases the impatient firms' deviation profits at a faster rates. Thus the price increase during the transition phase may be tied to a first-order condition which trades off the marginal gain of patient firms and the marginal deterrence cost (due to the marginal increase of deviation profit). This is the effect explaining gradualism in Kartal (2016), and we claim that considering a more general competitive setting would reinforce the first effect, to produce a smooth transition phase.

[16]For instance, in a Hotelling model, the collusive symmetric profit is linear in the price, while the deviation profit is quadratic and convex in the rival price (so that the difference is concave). In a linear Cournot model, the symmetric collusive profit is concave in q, but the deviation profit is convex in the rival's quantity.

7 Conclusion

We analyze a repeated Bertrand pricing game where one firm is privately informed as to its respective discount rate. We analyze the game from the separation period. In particular we characterize the payoff frontier after the separation period. The results show that patient firms employ gradually increasing prices. This induces low type firms to reveal their type early while giving the Pareto optimal payoffs to patient firms. We give the best speed for the transition phase and we associate a length to it. In the last section we construct the Pareto frontier from period 0 for a subclass of strategies: money-burning strategies. We exhibit the fact that firm A is free-riding : as sustainability conditions require the industry profit to be shared between the two firms, but the deterrence cost is borne only by firm $B(\delta_H)$. Hence firm $B(\delta_H)$ underinvests in deterrence and the transition phase is not "bang-bang".

We believe our results should apply when firm A and $B(\delta_H)$ don't have the same discount rate (provided collusion is still possible). The next step is to characterize the frontier from the separation period. In doing this we have also analyzed the game using public randomization device which smoothen the transition between the two types of strategy. Eventually, a important questions to solve are is when it is optimal to separate types, and whether separation in mix strategies improves the payoffs. In pure strategies, it can be shown that firm A is always better-off when separation occurs in period 0, and we believe it is the case for firm $B(\delta_H)$ as well.

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A

Proof of proposition 3.1

We split the proof in two cases. Either all $\Pi_{A,t} + \Pi_{B,t} < r$ in this case we show we can increase all $\Pi_{A,0} + \Pi_{B,0}$. Or there is a t for which $\Pi_{A,t+1} + \Pi_{B,t+1} = r$ and $\Pi_{A,t} + \Pi_{B,t} < r$, where we show that $\Pi_{A,t} + \Pi_{B,t}$ can be increased without decreasing any firms profit.

i) Assume for all $t \geq 0$, $\Pi_{A,0} + \Pi_{B,0} < r$. Consider the following $\{\pi'_{A,t}, \pi'_{B,t}\}_{t\geq 0}$, for a $\epsilon > 0$:

$$\pi'_{A,t} = \pi_{A,t}(1+\epsilon)$$
$$\pi'_{B,t} = \pi_{B,t}(1+\epsilon)$$

Notice that $\Pi'_{k,t} = (1+\epsilon)\Pi_{k,t}$ for all t and $k = A, B$, so that sustainability conditions are unaffected. Take ϵ small enough so that $\Pi'_{A,t} + \Pi'_{B,t} < r$ for all t. Moreover, both path can be made arbitrarily close taking a small enough ϵ:

$$\begin{aligned} \|\{\pi'_{A,t}, \pi'_{B,t}\}_{t\geq 0} - \{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}\|_\infty &= \epsilon \|\{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}\|_\infty \\ &\leq \epsilon r \end{aligned}$$

But $\Pi'_{A,0} = (1+\epsilon)\Pi_{A,0} > \Pi_{A,0}$ and $\Pi'_{B,0} = (1+\epsilon)\Pi_{B,0} > \Pi_{B,0}$, and so profit of both firms strictly increase.

ii) Let $\Pi_{A,t} + \Pi_{B,t} < r$ and $\Pi_{A,t+1} + \Pi_{B,t+1} = r$. Consider the following $\{\pi'_{A,t}, \pi'_{B,t}\}_{t\geq 0}$, for a $\epsilon > 0$:

$$\begin{aligned} &\pi'_{A,t} + \pi'_{B,t} = \pi_{A,t} + \pi_{B,t} + \epsilon && \\ &\pi'_{A,s} + \pi'_{B,s} = \pi_{A,s} + \pi_{B,s} && \forall\ s \neq t \\ &\pi'_{A,s} = \pi_{A,s} && \forall\ s < t \\ &\pi'_{A,s} = \pi_{A,s} && \forall s > t \end{aligned}$$

For $\pi'_{A,t}$:
(a) If S(B,t) slacks then set $\pi'_{A,t} = \pi_{A,t} + \epsilon$.
(b) If S(B,t) binds then set $\pi'_{A,t} = \pi_{A,t}$.

With ϵ small enough so that $S(B,t)'$ and $S(A,t)'$ still hold, and $\pi_{A,t} + \pi_{B,t} + \epsilon < r$. Because $\Pi'_{A,t} + \Pi'_{B,t} < \Pi'_{A,t+1} + \Pi'_{B,t+1}$, at least one sustainability condition at t slacks.
Moreover, both path can be made arbitrarily close taking a small enough ϵ:

$$\|\{\pi'_{A,t}, \pi'_{B,t}\}_{t\geq 0} - \{\pi_{A,t}, \pi_{B,t}\}_{t\geq 0}\|_\infty = \epsilon$$

In case (a): $\Pi'_{A,t} = \Pi_{A,t} + (1-\delta_H)\epsilon > \Pi_{A,t}$ and $\Pi'_{B,t} = \Pi_{B,t}$. Hence firm A's profit strictly increases while firm B's profit does not decrease.

In case (b): $\pi'_{B,t} = \pi_{B,t} + \epsilon$ and so, $\Pi'_{B,t} = \Pi_{B,t} + (1-\delta_H)\epsilon > \Pi_{B,t}$ and $\Pi'_{A,t} = \Pi_{A,t}$. Hence firm B's profit strictly increases while firm A's profit does not decrease.

Hence locally, at least one firm's profit can be strictly increased while the other does not decrease. ∎

B

Proof of lemma 4.1

i) Assume that $p_{A,t} < 0$.
i.a) $p_{A,t} < p_{B,t}$, or $p_{B,t} > p_{A,t}$. Consider a slight increase in $p_{A,t}$ (so that the ordering of price

remains). Sustainability conditions are relaxed: the best deviation at t for B yields at least 0 (the minmax), and firm's A profit weakly increases. Moreover, equilibrium conditions for firm δ_L are unchanged: payoff of undercutting (revealing) is still 0 (minmax), while the mimicking payoff is unchanged.
i.b) $p_{A,t} = p_{B,t}$. Now consider a slight increase in both prices (keeping the equality), and a variation in α_t so that $\pi_{B,t}$ stays constant (α_t decreases slightly). The same arguments apply, equilibrium conditions are unchanged or relaxed, but firm's A profit strictly increases.

ii) Assume that $p_{B,t} < 0$ for $t \geq 1$, (wlog consider $p_{A,t} = 0$).
Consider a slight increase in $p_{B,t}$ so that : $d\pi_{B,t-1} = -\delta_L dp_{B,t}$, keeping $\pi_{A,t}$ constant.

We pause to explicit the effect on α_{t-1}, and $(p_{B,t-1}, p_{A,t-1})$. If $p_{A,t-1} > p_{B,t-1}$ (or equality with $\alpha_0 = 0$) then only $p_{B,t-1}$ varies. If $p_{A,t-1} < p_{B,t-1}$ then a) set $p'_{B,t-1} = p_{A,t-1}$ (relaxes S(A,t)) and b) decrease the market price keeping $\pi_{A,t-1}$ constant (increasing α_0, note that in this case by point *i*) prices are positive).

Effect on equilibrium conditions:
Sustainability conditions are relaxed: in the case where $p_{A,t}$ was higher, the best deviation for B is unchanged ($p_{A,t-1}$), and for A at t best deviation at 0 (so unchanged) and at $t-1$ A the best deviation profit weakly decreases (0 or $p_{B,t-1}$).
In the second case, firm A sustainability condition are relaxed : market price decreases in $t-1$, and best deviation at 0 in t still. For firm B, note in $t-1$ the continuation profit of B increases, despite a loss in market share at this period.

Equilibrium conditions for δ_L firm. First remark that undercutting in t provides a payoff of 0, hence the payoff of mimicking until t changes only via the $t-1$ period. We will show that payoff of mimicking until $t-1$, and until $s \geq t+1$ is weakly lower as well.
For $s \geq t+1$, remark that $d\pi_{B,t-1} = \delta_L dp_{B,t}$, hence this profit is unchanged.
For $t-1$ (and so also for t) : the market price in $t-1$ decrease and in the first case $p_{A,t-1}$ is unchanged, hence δ_L profit's is weakly. lower.

iii)Consider period $t \geq 0$, where $p_{A,t}$ and $p_{B,t}$ are positive.
iii.a) for $t \geq 1$: consider the following variation $p'_{A,t} = p'_{B,t} = \min\{p_{A,t}, p_{B,t}\}$, α_t is constant. Profits remains constant, and deviation profits are lowered since both prices weakly decreases. Hence equilibrium conditions still hold, and profits are unchanged.

iii.b) For $t = 0$, the previous variation fails only when $p_{A,0} > p_{B,0}$, in which case the equilibrium payoff of δ_L is strictly lower.
In that case consider another variation: $p'_{B,0} = p_{A,0}$, and $\alpha'_0 p'_{B,0} = p_{B,0}$. In this case profit of firm B δ_H and δ_L are unchanged, as well as incentives to mimic. Profits of firm A strictly increases since she obtains profit in period 0. Because the market price has increased by the same amount of A's profit, her equilibrium condition in 0 is unchanged. ∎

C

Proof of lemma Proposition 5.1

Assume there is a sequence of states $\{\Pi_{L,t}\}_{t\geq 1}$ satisfying *i*) and *ii*). Using the law of motion we have, for any $t \geq 1$: $\Pi_{L,1} = \sum_{s=1}^{t-1} \delta_L^{s-1} \pi_{B,s} + \delta_L^{t-1} \Pi_{L,t}$. Using the non-mimicking condition at t we have for any t:

$$\Pi_{L,1} \geq \sum_{s=1}^{t} \delta_L^{s-1} \pi_{B,s} + \delta_L^{t-1} \pi_{A,t}$$

Hence, the path yields a mimicking profit lower than $\Pi_{L,1}$.

Conversely, we construct a sequence of states $\{\Pi_{L,t}\}_{t\geq 1}$ satisfying $i)$ and $ii)$, from a given sustainable path. Given $\Pi_{L,1}$ and the path, we can directly construct the sequence of states according to the law of motion. One can show each state satisfies non-mimicking. ∎

Proof of proposition 5.2

i) Recall $\mathcal{C}(\Pi_{L,t})$ is included in a compact set because $\pi_{k,t} \geq 0$ and $\pi_{A,t} + \pi_{B,t} \leq r$. The additional constraints enter with weak inequality hence $\mathcal{C}(\Pi_{L,t})$ is a closed set included in a compact set, therefore compact. $(0,0) \in \mathcal{C}(\Pi_{L,t})$ for all $\Pi_{L,t} \geq 0$ using the Nash equilibrium. Thus $\mathcal{C}(\Pi_{L,t})$ is a non empty compact set.

ii)By construction: $\mathcal{C}(\Pi_{L,t}) = (1-\delta_H)\mathcal{I}(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})) + \delta_H \mathcal{C}(\Pi_{L,t+1})$. We prove that:

$$F\left\{(1-\delta_H)\mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right) + \delta_H \mathcal{C}(\Pi_{L,t+1})\right\} = F\left\{(1-\delta_H)\mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right) + \delta_H F(\Pi_{L,t+1})\right\}$$

Let $(\Pi_{A,t+1}, \Pi_{B,t+1}) \in \mathcal{C}(\Pi_{L,t+1})\backslash F(\Pi_{L,t+1})$. By definition, there exists $(\Pi'_{A,t+1}, \Pi'_{B,t+1}) \in F(\Pi_{L,t+1})$ for which $(\Pi'_{A,t+1}, \Pi'_{B,t+1}) \geq (\Pi_{A,t+1}, \Pi_{B,t+1})$ and $(\Pi'_{A,t+1}, \Pi'_{B,t+1}) \neq (\Pi_{A,t+1}, \Pi_{B,t+1})$. At this point we have: $\mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right) \subset \mathcal{I}\left(\Pi_{L,t}, (\Pi'_{A,t+1}, \Pi'_{B,t+1})\right)$, since sustainability conditions are relaxed. Hence any point in the set
$(1-\delta_H)\mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right) + \delta_H \mathcal{C}(\Pi_{L,t+1})\backslash F(\Pi_{L,t+1})$ can be Pareto dominated by a point in $(1-\delta_H)\mathcal{I}\left(\Pi_{L,t}, (\Pi_{A,t+1}, \Pi_{B,t+1})\right) + \delta_H F(\Pi_{L,t+1})$. Therefore the above equality is true. ∎

D

We first proof a useful lemma

Lemma D.1 *Assume $\underline{L}(t)$ is binding, and S(B,t) holds.*
For all $i \leq t-1$, if $\underline{L}(t-i)$ holds then S(B,t-i) is slacking.

Proof $\underline{L}(t)$ is binding: $\Pi_L = \sum_{s=1}^{t} \delta_L^{s-1}\pi_{B,s} + \delta_L^t \pi_{A,t}$. Subtracting to $\underline{L}(t-i)$ we have: $\pi_{A,t-i} + \pi_{B,t-i} \leq \sum_{s=t-i}^{t} \delta_L^{s-t+i}\pi_{B,s} + \delta_L^t \pi_{A,t}$. Multiplying by $1-\delta_H$ and using both S(B,t) and $\delta_L < \delta_H$ we have: $(1-\delta_H)(\pi_{A,t-i} + \pi_{B,t-i}) < (1-\delta_H)\sum_{s=t-i}^{t} \delta_H^{s-t+i}\pi_{B,s} + \delta_H^i \Pi_{B,t}$. Hence : $(1-\delta_H)(\pi_{A,t-i} + \pi_{B,t-i}) < \Pi_{B,t-i}$.

iii) For $\Pi'_{L,t} \geq \Pi_{L,t}$ constraints are relaxed, thus $C(\Pi_{L,t}) \subset C(\Pi'_{L,t})$, and therefore there does not exist any point in $C(\Pi_{L,t})$ which Pareto dominates all points in $C(\Pi'_{L,t})$. ∎

Proof of lemma 5.2

i) When $\Pi_{L,t} = 0$ the only path satisfying the constraints is the Nash equilibrium path. Hence, $(\Pi_{A,t}, \Pi_{B,t}) = (0,0)$ and all $\underline{\text{L}}$ are binding at 0. Conversely, all $\underline{\text{L}}$ binding means that the path is sustainable for firm (δ_L). Hence this path is the Nash equilibrium path and thus $\Pi_{L,t} = 0$

ii) $\Pi_{L,t} > r$ the complete information frontier is achievable: take a path such that for all $s \geq 0$ $\pi_{A,t+s} + \pi_{B,t+s} = r$. Sustainability conditions are equivalent to:

$$(1-\delta_H)r \leq \Pi_{B,t+s} \leq \delta_H r$$

Implement $(1-\delta_H)r\Pi_{B,t} \leq \delta_H r$ using the constant per period profit $\pi_{B,t+s} = \Pi_{B,t}$ and we have:

$\Pi_{L,t} > r$	$\underline{\text{L}}$(t) is slacking
$\Pi_{L,t+1} > \dfrac{1-\delta_H}{\delta_L} > r$	$\underline{\text{L}}$(t+1) is slacking
$\Pi_{L,t+s+1} > r$	by induction, $\underline{\text{L}}$(t+s) is slacking for all $s \geq 1$

Conversely, assume all L(t+s) are slacking for $s \geq 0$. The problem is thus the one of complete information and so $(\Pi_{A,t}, \Pi_{B,t})$ is on the frontier only if $\Pi_{A,t} + \Pi_{B,t} = r$. For L(t+s) to be slacking we need $\Pi_{L,t} > r$.

iii) Assume, $0 < \Pi_{L,t} \leq r$.

Step 1: we prove that if $0 \leq \Pi_{L,t} \leq r$ then L(t) is binding. For $\Pi_{L,t} = 0$ we know it is true from i). We focus on $0 < \Pi_{L,t} \leq r$.

Suppose not: L(t) is slacking (which is possible because $\Pi_{L,t} > 0$), so that we have $\pi_{A,t} + \pi_{B,t} < \Pi_{L,t} \leq r$
Case 1: all future L are slacking as well. Locally the problem is equivalent to the complete information problem, where we have $\Pi_{A,t} + \Pi_{B,t} < r$, thus using proposition 3.1 this cannot be optimal.
Case 2: Assume L(t+i) binds for some $i \geq 1$. Using lemma 5.3 S(B,t) slacks and so $\pi_{A,t}$ can be increased without violating any constraints, a contradiction.

Therefore, L(t) is binding whenever $0 \leq \Pi_{L,t} \leq r$.

Observation 1: If L(t) is slacking then $\Pi_{L,t} > r$ (contrapositive of step 1), hence applying ii) L(t+s) is slacking for all $s \geq 0$.

Observation 2: From i), since $\Pi_{L,t} > 0$, there exists $i \geq 0$ such that L(t+i) is slacking. Hence, using observation 1 there exists an $i \geq 0$ where $\Pi_{L,t+i} > 0$ and so L(t+s) is slacking for all $s \geq i$.

To sum up, we know that for $0 < \Pi_{L,t} \leq r$, L(t) is binding, and there exists an $i \geq 1$ such that L(t+s) is slacking for all $s \geq i$. Take the smallest i and call it l. Because l is defined from the smallest i we have that L(t+s) is binding for $s < l$. Therefore, for all $0 < \Pi_{L,t} \leq r$ there exists $l \geq 1$ for which L(t+s) binds for $s < l$ and slacks for $s \geq l$.

For the converse: using i) then $\Pi_{L,t} > 0$ (at least one condition is slacking). Using ii) then $\Pi_{L,t} \leq r$ (at least one condition is binding). ∎

Proof of lemma 5.3

i) Suppose not: $\pi_{A,t} + \pi_{B,t} > (\delta_L + \delta_H)(\pi_{A,t+1} + \pi_{B,t+1})$.

Step 1: we show $S(A, t+1)$ is slacking.

Because $0 < \Pi_{L,t+1} \leq r$, using lemma 5.3 iii): L(t+1) is binding. Thus L(t) binds as well. Hence we have the following equality: $\pi_{A,t} + \pi_{B,t} = \pi_{B,t} + \delta_L(\pi_{A,t+1} + \pi_{B,t+1})$. Using $\pi_{A,t} + \pi_{B,t} > (\delta_L + \delta_H)(\pi_{A,t+1} + \pi_{B,t+1})$ we have:

$$\pi_{B,t} > \delta_H(\pi_{A,t+1} + \pi_{B,t+1}) \quad (1)$$

From S(A,t):

$$\begin{aligned} & \Pi_{A,t} \geq (1 - \delta_H)(\pi_{A,t} + \pi_{B,t}) \\ \iff & \delta_H \Pi_{A,t+1} \geq (1 - \delta_H)\pi_{B,t} \\ \implies & \delta_H \Pi_{A,t+1} > (1 - \delta_H)\delta_H(\pi_{A,t+1} + \pi_{B,t+1}) \quad \text{using (1)} \\ \iff & \Pi_{A,t+1} > (1 - \delta_H)(\pi_{A,t+1} + \pi_{B,t+1}) \quad \text{S(A,t+1) is slacking} \end{aligned}$$

Therefore S(A,t+1) is slacking.

Step 2: We show there is a variation which is Pareto improving.

Consider the following variation, for $d\pi_{B,t} < 0$.

$$-d\pi_{B,t} = \delta_L d\pi_{B,t+1}$$

$\Pi_{A,t}$ is unaffected. $\underline{\text{L}}$(t) is relaxed, and $\underline{\text{L}}$(t+s) are unaffected for $s \geq 1$. S(A,t) is relaxed as $\pi_{B,t}$ is reduced. Because $\underline{\text{L}}$(t+1) binds, S(B,t) is slacking hence reducing $\pi_{B,t}$ does not violate S(B,t). Using step 1 S(A,t+1) slacks hence increasing $\pi_{B,t+1}$ is admissible for a small variation.
Eventually, we need to check $\pi_{B,t} > 0$ for the variation to be admissible. But recall (1): $\pi_{B,t} > \delta_H(\pi_{A,t+1} + \pi_{B,t+1}) = \delta_H \Pi_{L,t+1} > 0$.

Therefore the variation is admissible for small $d\pi_{B,t} < 0$. $\Pi_{A,t}$ is unaffected while $\Pi_{B,t}$ increases strictly:

$$\begin{aligned} d\Pi_{B,t} &= (1-\delta_H)(d\pi_{B,t} + \delta_H d\pi_{B,t+1}) \\ &= (1-\delta_H)(\delta_H - \delta_L) d\pi_{B,t+1} > 0 \end{aligned}$$

Hence we have a contradiction, the path considered cannot yield optimal payoff and thus $\pi_{A,t}+\pi_{B,t} \leq (\delta_H + \delta_L)(\pi_{A,t+1} + \pi_{B,t+1})$.

ii) $\delta_L(\pi_{A,t+1} + \pi_{B,t+1}) \leq \pi_{A,t} + \pi_{B,t}$. $\underline{\text{L}}$(t) and $\underline{\text{L}}$(t+1) are binding implies $\pi_{A,t} + \pi_{B,t} = \pi_{B,t} + \delta_L(\pi_{A,t+1} + \pi_{B,t+1})$ because $\pi_{B,t} \geq 0$ we have: $\pi_{A,t} + \pi_{B,t} \geq \delta_L(\pi_{A,t+1} + \pi_{B,t+1})$. ∎

Proof of lemma 5.4

ii)S(A,t+s) is binding for $0 \leq s \leq l-1$. So $(1-\delta_H)\pi_{B,t+s} = \delta_H \Pi_{A,t+s+1}$. From $\underline{\text{L}}$(t+s) and the low of motion binding we have $\pi_{A,t+s} + \pi_{B,t+s} = \Pi_{L,t+s}$ and $\Pi_{L,t+s+1} = \frac{\pi_{A,t+s}}{\delta_L}$. Plug in $\underline{\text{L}}$(t+s) we have: $(1-\delta_H)(\Pi_{L,t+s} - \delta_L \Pi_{L,t+s+1}) = \delta_H \Pi_{A,t+s+1}$. Because $\underline{\text{L}}$(t+s+1) is binding, for S(A,t+s+1) to hold we need: $(1-\delta_H)\Pi_{L,t+s+1} \leq \Pi_{A,t+s+1}$. Hence a necessary condition for S(A,t+s+1) to hold is $\frac{\Pi_{L,t+s} - \delta_L \Pi_{L,t+s+1}}{\delta_H} \geq \Pi_{L,t+s+1} \iff \Pi_{L,t+s} \geq (\delta_H + \delta_L)\Pi_{L,t+s+1}$. Using lemma 5.4 (lower bound on the speed) we have $\Pi_{L,t+s} = (\delta_H + \delta_L)\Pi_{L,t+s+1}$, and so S(A,t+s+1) binds. Moreover we have $\pi_{A,t+s} = \frac{\delta_L \Pi_{L,t+s}}{\delta_L + \delta_H}$, $\pi_{B,t+s} = \frac{\delta_H \Pi_{L,t+s}}{\delta_L + \delta_H}$.

iii) For the contradiction assume S(A,t+s) slacks for a $1 \leq s \leq l-1$, and that $\pi_{B,t+s-1} > 0$.
Consider the following variation:

$d\pi_{B,t+s-1} + \delta_L d\pi_{B,t+s} = 0$	future $\underline{\text{L}}$(t+s) unchanged
$d\pi_{B,t+s-1} = -d\pi_{A,t+s-1}$	past and present $\underline{\text{L}}$(t+s-1) unchanged
$d\pi_{A,t+s-1} = \delta_L(d\pi_{A,t+s} + d\pi_{B,t+s})$	$\underline{\text{L}}$(t+s) unchanged

S(B,t+s) is slacking by lemma 5.2, and s(A,t+s) is slacking by assumption. Therefore the variation is admissible for small enough $-d\pi_{B,t+s-1} > 0$. We have:

$$\begin{aligned} d\pi_{A,t+s} &= 0 \\ d\pi_{A,t+s-1} &= -d\pi_{B,t+s-1} \\ d\pi_{B,t+s} &= -\frac{\delta_H - \delta_L}{\delta_H \delta_L} \end{aligned}$$

Therefore the variation of profits is:

$$\begin{aligned} d\Pi_{B,t+s-1} &= -\frac{\delta_H - \delta_L}{\delta_L} d\pi_{B,t+s-1} \\ d\Pi_{A,t+s-1} &= -d\pi_{B,t+s-1} \end{aligned}$$

Hence for $-d\pi_{B,t+s-1} > 0$ both profits strictly increases, hence we have a contradiction.

Hence when S(A,t+s) slacks we have $\pi_{B,t+s-1} = 0$, and thus using $\underline{\text{L}}$(t+s) and $\underline{\text{L}}$(t+s-1) binding we can compute: $\pi_{A,t+s} = \Pi_{L,t+s}$, $\pi_{B,t+s} = 0$, $\Pi_{L,t+s+1} = \frac{\Pi_{L,t+s}}{\delta_L}$.

i) Point ii) proof, implies that i) is true for any $0 \leq s \leq l-2$, whenever $l \geq 2$. We thus have to prove i) for $s \geq l-1$, or $s \geq 0$ if $l = 0$.

Case 1: $s \geq l$. Continuation payoffs are the one of the complete information game, hence the payoffs can be achieved by stationary paths. From which either all S(A,s) are binding, or all S(A,s) are slack for all $s \geq l$.

Case 2: $s = l-1$, $l \geq 1$.

Assume S(A,l-1) is binding. We have:

$$\begin{aligned} &\Pi_{B,t+l-1} = \delta_H(\Pi_{A,t+l} + \Pi_{B,t+l}) \\ \iff &\Pi_{B,t+l-1} = \delta_H r \end{aligned}$$

Recall the highest payoff a firm can claim in the complete information game equilibrium payoffs is $\delta_H r$. That is $\Pi_{B,t+s} \leq \delta_H r$ for all $s \geq l$. Hence to maintain the first equality, by construction $\Pi_{B,t+s} = \delta_H r$ for all $s \geq l$ and so S(A,t+s) is binding for all $s \geq l$. ∎

Computations for the characterization

Given $\Pi_{L,t}$, We will show to all optimal path indexed by (l,l), there is a unique corresponding vector payoff for each (l,l). The path is identified (in the sens that S(A,.) and $\underline{\text{L}}$ binding leads to an identified system of equations) for $s \leq l-1$. Moreover, we have S(A,l-1) binding:

$$\begin{aligned} \Pi_{B,t+l-1} &= \delta_H(\Pi_{B,t+l} + \Pi_{A,t+l}) \\ &= \delta_H r \qquad \text{Complete information industry payoff} \end{aligned}$$

Hence,

$$\begin{aligned} \Pi_{A,t+l-1} &= (1-\delta_H)\Pi_{L,t+l-1} + \delta_H r - \delta_H r \\ &= (1-\delta_H)\frac{\Pi_{L,t}}{(\delta_L+\delta_H)^{l-1}} \end{aligned}$$

Therefore

$$\begin{aligned} F_{l,l}(\Pi_{L,t}) &= (1-\delta_H)\sum_{s=0}^{l-2}\left(\frac{\delta_H}{\delta_H+\delta_L}\right)^s \Pi_{L,t}\{(\frac{\delta_L}{\delta_H+\delta_L}, \frac{\delta_H}{\delta_H+\delta_L})\} + \delta_H^{l-1}\{(\delta_H r, (1-\delta_H)\frac{\Pi_{L,t}}{(\delta_H+\delta_L)^{l-1}})\} \\ &\equiv \{\Pi(l,l)\} \end{aligned}$$

with the sum being 0 by convention if $l = 1$.

This point satisfies $\underline{\text{L}}$(t+l) being slack and $\underline{\text{L}}$(t+l-1) binding if:

$$\begin{aligned}
&\Pi_{L,t+l-1} \leq r \\
&\iff \Pi_{L,t} \leq (\delta_L + \delta_H)^{l-1} r
\end{aligned}$$

$$\begin{aligned}
&\Pi_{L,t+l} > r \\
&\iff \Pi_{L,t+l-1} - \delta_H r > \delta_L r \\
&\iff \Pi_{L,t} > (\delta_H + \delta_L)^l r
\end{aligned}$$

Therefore, if $F_{l,l}(\Pi_{L,t}) \neq \emptyset$ then $(\delta_L + \delta_H)^l r < \Pi_{L,t} \leq (\delta_L + \delta_H)^{l-1}$ (which gives i)).

for ii) The unique degree of freedom is in period $t + l - a - 1$, in the allocation of $\Pi_{L,t+l-a-1}$ between the two firms. Hence, profits indexed by (l, a) given a $\Pi_{L,t}$ are located on a segment.

$\underline{\text{L}}$(t+l) being slack and $\underline{\text{L}}$(t+l-1) binding give an upper bound and lower bound on $\pi_{B,t+l-a-1}$:

$$\begin{aligned}
&\Pi_{L,t+l-1} \leq r \\
&\iff \frac{\Pi_{L,t+l-a-1} - \pi_{B,t+l-a-1}}{\delta_L(\delta_L + \delta_H)^{a-1}} \leq r \\
&\iff \pi_{B,t+l-a-1} \geq \frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^{a-1}\delta_L r
\end{aligned}$$

$$\begin{aligned}
&\Pi_{L,t+l} > r \\
&\iff \Pi_{L,t+l-1} - \delta_H r > \delta_L r \\
&\iff \pi_{B,t+l-a-1} < \frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^{a}\delta_L r
\end{aligned}$$

Moreover S(A,t+l-a-1) being slack and $\pi_{B,t+l-a-1} \geq 0$ give an upper bound and lower bound on $\pi_{B,t+l-a-1}$:

$$\begin{aligned}
&\pi_{B,t+l-a-1} < \frac{\delta_H \Pi_{L,t+l-a-1}}{\delta_L + \delta_H} \\
&\iff \pi_{B,t+l-a-1} < \frac{\delta_H \Pi_{L,t}}{(\delta_L + \delta_H)\delta_L^{l-a-1}}
\end{aligned}$$

$$\pi_{B,t+l-a-1} \geq 0$$

All the constraints can hold whenever

$$\begin{aligned}
&0 < \frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^{a}\delta_L r \\
&\iff \Pi_{L,t} > \delta_L^{l-a}(\delta_H + \delta_L)^a r
\end{aligned}$$

$$\begin{aligned}
&\frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^{a-1}\delta_L r < \frac{\delta_H \Pi_{L,t}}{(\delta_L + \delta_H)\delta_L^{l-a-1}} \\
&\iff \Pi_{L,t} < \delta_L^{l-a-1}(\delta_L + \delta_H)^a r
\end{aligned}$$

which gives ii) To sum up, for a path (l, a) to be optimal we necessarily need :

$$\delta_L^{l-a}(\delta_H + \delta_L)^a r < \Pi_{L,t} < \delta_L^{l-a-1}(\delta_L + \delta_H)^a r$$

$$\begin{aligned}
&\max\left\{0, \frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^{a-1}\delta_L r\right\} \leq \pi_{B,t+l-a-1} \\
&< \min\left\{\frac{\delta_H \Pi_{L,t}}{(\delta_L + \delta_H)\delta_L^{l-a-1}}, \frac{\Pi_{L,t}}{\delta_L^{l-a-1}} - (\delta_H + \delta_L)^a \delta_L r\right\}
\end{aligned}$$

iii) $a = 0$
For all $s \leq l - 2$ we have $\pi_{B,s} = 0$, and $\Pi_{L,s} = \frac{\Pi_{L,t}}{\delta_L^{s-t}}$

$\underline{L}(t + l)$ being slack and $\underline{L}(t + l - 1)$ binding gives conditions on $\Pi_{L,t}$:

$$\begin{aligned}
&\Pi_{L,t+l} > r \iff \Pi_{L,t+l-1} - \pi_{B,t+l-1} > \delta_L r \\
&\iff \Pi_{L,t} > \delta_L^l r \\
&\Pi_{L,t+l-1} \leq r \iff \Pi_{L,t} \leq \delta_L^{l-1} r
\end{aligned}$$

which gives iii).

Computing the slope

In each segments we compute the slope (on $(\Pi_{A,t}, \Pi_{B,t})$ plane). $\equiv \frac{d\Pi_{B,t}}{d\Pi_{A,t}}$

For a segment indexed by $(l, l - 1)$. Consider the following variation, increasing the market share for firm A in period t:

$$\begin{aligned}
&-d\pi_{B,t} = d\pi_{A,t} \\
&d\Pi_{L,t+s} = \frac{d\pi_{A,t}}{\delta_L(\delta_L + \delta_H)^{s-1}} && \forall 1 \leq s \leq l - 1 \\
&d\pi_{B,t+s} = \frac{\delta_H}{\delta_L + \delta_H} \frac{d\pi_{A,t}}{\delta_L(\delta_L + \delta_H)^{s-1}} && \forall 1 \leq s \leq l - 2 \\
&d\pi_{A,t+s} = \frac{\delta_L}{\delta_L + \delta_H} \frac{d\pi_{A,t}}{\delta_L(\delta_L + \delta_H)^{s-1}} && \forall 1 \leq s \leq l - 2 \\
&d\Pi_{B,t+l-1} = 0 \\
&d\Pi_{A,t+1} = (1 - \delta_H)d\Pi_{L,t+1} = (1 - \delta_H)\frac{d\pi_{A,t}}{\delta_L}
\end{aligned}$$

Therefore we have, for firm B:

$$\begin{aligned}
d\Pi_{B,t} &= (1 - \delta_H)\left(-d\pi_{A,t} + \frac{\delta_H}{\delta_L}\sum_{s=1}^{l-2}\left(\frac{\delta_H}{\delta_L + \delta_H}\right)^s d\pi_{A,t}\right) \\
&= (1 - \delta_H)d\pi_{A,t}\left[\frac{\delta_H + \delta_L}{\delta_L}\left(\frac{\delta_H - \delta_L}{\delta_L} - \frac{\delta_H}{\delta_L}\left(\frac{\delta_H}{\delta_L + \delta_H}\right)^{l-1}\right)\right]
\end{aligned}$$

And for firm A:

$$\begin{aligned}
d\Pi_{A,t} &= (1 - \delta_H)\left(d\pi_{A,t} + \delta_H\frac{d\pi_{A,t}}{\delta_L}\right) \\
&= (1 - \delta_H)\frac{\delta_L + \delta_H}{\delta_L}d\pi_{A,t}
\end{aligned}$$

And so the slope is :

$$\frac{d\Pi_{B,t}}{d\Pi_{A,t}}(l,l-1) = \frac{\delta_H}{\delta_L}\left(\frac{\delta_H-\delta_L}{\delta_H} - \left(\frac{\delta_H}{\delta_L+\delta_H}\right)^{l-1}\right)$$

E

We proof a clarifying lemma first :

Lemma E.1 *In a path yielding Pareto optimal payoffs, with $l > 1$ we have:*

$$\textit{If} \quad \pi_{B,0} < 0, \quad \textit{then } S(A,t) \textit{binds } \forall\ 1 \leq t \leq l-1$$

Proof Remark that here l is the last period where non-mimicking condition binds, since the "optimal instrument problem" starts in period 1.
Moreover, because the path from period 1 is a solution of the optimal instrument problem we have that for $t > 1$. Hence it is sufficient to prove that $S(A,1)$ binds.
Suppose not: $S(A,1)$ is slack.
S(A,1) is slack. Consider the following variation :

$$d\pi_{B,0} = -\frac{\pi_{B,1}}{\delta_L}$$
$$d\pi_{B,0} < 0$$

Remark that $\Pi_{L,1} = -\frac{\pi_{B,0}}{\delta_L}$ and so non-mimicking condition still hold in period 1, while future non mimicking conditions are unaffected. Also, S(A,1) is slack. And, because $l > 1$: $\Pi_{L,1} < r$, hence for a variation small enough it is admissible. While:

$$d\Pi_{A,0} = 0$$
$$d\Pi_{B,0} = -d\pi_{B,0}\frac{\delta_H-\delta_L}{\delta_L} > 0$$

∎

This is intuitive: if firm $B(\delta_H)$ is willing to deter firm $B(\delta_L)$ from mimicking, the first period is the best one to do it. Hence, in the frontier for money burning strategy, whenever $l > 1$ then it must be a (l,l) path following period 0.

Thus, when $l > 1$ the only degree of freedom is $p_{B,0}$, does reducing $p_{B,0}$ to increase future profit compensate for the cost for firm $B(\delta_H)$. This trade-off is the same as in section 5. Henceforth, the frontier stops when the path length reaches $\overline{a}$, this entails the upper bound on $p_{B,0}$, and the proposition follows.

For the close form formulas:

$$\text{For } -\delta_L(\delta_L+\delta_H)^{l-1} \leq p_{B,0} < -\delta_L(\delta_L+\delta_H)^l r$$
$$\Pi_{A,0}(p_{B,0},0) = -(1-\delta_H)\delta_H\frac{p_{B,0}}{\delta_L}$$
$$\Pi_{B,0}(p_{B,0},0) = (1-\delta_H)p_{B,0}\left[\frac{(\delta_H+\delta_L)\delta_H}{\delta_L^2}\left(\left(\frac{\delta_H}{\delta_L+\delta_H}\right)^l - \frac{\delta_H-\delta_L}{\delta_H}\right)\right] + \delta_H^l r$$