On the Relation of Country Size to the Form of Government

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Abstract

A mechanism governing the territory exchange between groups of people such as countries or municipalities is proposed based on land trading with approval of both sides under particular voting rule (majority rule, unanimous rule, etc.). Voting rules can model different forms of government such as monarchy, oligarchy, and democracy. Under these forms of government the influence of wealth inequality on the territory exchange result is studied. Conquest of the territory is considered as a special case of trading, when the buyer country pays not to the seller country but to the army which conquer territory for the buyer. Conditions at which countries prefer to trade land rather than fight for it are found using game theoretic approach.

Introduction

People form groups in order to benefit from it no matter the type of the group whether it firm, labor union or country. The main questions here are how the decision making mechanism adopted by the group influences its wellbeing which can be measured by a social welfare function or by the amount of resources accumulated by the group and when these resources are accumulated peacefully. We study these questions on the example of two neighboring countries which use their territories as a production factor (capital). Country can trade part of its territory for the part of other country production making such decision according to the voting rules corresponding to the forms of governments in the countries. So here countries look more like firms competing for the limited resources.

Private good produced in the country is in general not equally distributed among the citizens besides that part which is given to the citizens of neighboring country for their decision to cede the part of the territory of their country. Thus, we study the problem with generally nonequal wealth distribution among the agents with endogenous income.

In the presented study we try to avoid some not very realistic assumptions such as exogenous income, uniform distribution of the territory among agents and free will or full mobility of every agent when she independently decide which country to join. It is typical in literature to use such assumptions for allocation of public facilities (Cremer, Kerchove, & Thisse, 1985) or for the similar problem of the equilibrium size and number of nations on the continuum of uniformly distributed individuals (Alesina & Spolaore, 1997), where each individual at the border between two countries can choose which country to join with her land. Alesina and Spolaore (1997) have also considered the coalition equilibrium as well as in (Bolton & Roland, 1997). Charles M. Tiebout in his paper (Tiebout, 1956) considered equilibrium where the consumer-voter is fully mobile and will move to one of the fixed number of communities where her preference pattern is best satisfied. This concept have been criticized by Bewley (Bewley, 1981). He argued that if one tries to generalize the rigorous version of Tiebout's theory in a number of interesting directions, then equilibria may no longer exist or may no longer be Pareto optimal. The existence proof of "strong Tiebout equilibrium" in (Greenberg & Weber, 1986) makes use of the notion of "consecutive games" which the authors introduce and show that for such games there always exists a partition with a nonempty core. Resent more general researches like (Haimanko, Le Breton, & Weber, 2004) also need special assumptions about the structure of the model in order to prove existence of equilibrium. It seems that perfect mobility and free will of agents make it difficult to find an equilibrium, so we will go without such assumptions. The more so because there are always some mobility constraints in the real world.

There are few historical examples of territory trade between countries. The biggest is the purchase of Alaska (1,717,854 km^2) from the Russian Empire by the United States of America in 1867. The Treaty of Petrópolis between Bolivia and Brazil, signed on November 11, 1903 gave Brazil the territory of Acre (191,000 km^2), in exchange for over

3,000 km^2 of Brazilian territory between the Abunâ and Madeira rivers, a monetary payment of two million British pounds, and a pledge of a rail-link between the Bolivian city of Riberalta and the Brazilian city of Porto Velho.

Nevertheless, one could argue that usually countries conquer land rather than buy it. We consider conquest of the territory as a special case of trading, when the buyer country pays not to the seller country but to the third player a mercenary army which conquer territory for the buyer unless the seller pays to the mercenary at least the same amount to keep its territory.

Which tactics would countries prefer to fight or to deal? The same dilemma has been studied (e. g. Grossman & Mendoza, 2001, 2004) in the economic theory of empire building using examples of the Roman and other empires, where three strategies where considered: Uncoerced Annexation, Coerced Annexation, and Attempted Conquest. In an *Uncoerced Annexation* the Romans would compensate the Barbarians sufficiently to induce the Barbarians to agree to the annexation of their country by Rome. In a *Coerced Annexation* the Romans would induce the Barbarians to agree to the annexation of their country under the threat that the Romans will attack and try to conquer the country. In an *Attempted Conquest* the Romans would attack the Barbarian country. In contrast with choice-theoretic explanation by Grossman and Mendoza we tackle this problem with game-theoretic model, were both players have the same strategic possibilities but could have different parameters. We are to find the condition at which countries would trade their territory rather than conquer it. We will try to correspond the results with terms Uncoerced Annexation, Coerced Annexation, and Attempted Conquest.

This work elaborates some of author's results in (Belyakov, 2007).

1 Mechanisms of border moving by land trade

We consider two neighboring countries $i = 1, 2$ as firms with production functions $f_i(S_i)$ depending on the countries territories S_i in the following way

$$f_i(S_i) \geq 0, \quad \frac{df_i(S_i)}{dS_i} > 0, \quad \frac{d^2 f_i(S_i)}{dS_i^2} < 0 \quad \text{for all } S_i \geq 0. \tag{1}$$

So the function is positive, strictly increasing and concave, and defined for the positive territory. Each country i is populated with constant number of citizens N_i which have their shares $\theta_{j\,i}$ of domestic production. For convenience we assume that $\theta_{j\,i}$ are in ascending order over index i and normalized, i.e.

$$\theta_{j\,i} \leq \theta_{j+1\,i} \quad \text{for all } j = 1, \ldots, N_i - 1, \quad \text{and } \sum_{j=1}^{N_i} \theta_{j\,i} = 1 \quad \text{for all } i = 1, 2.$$

The agents have linear utility with respect to the product which can be considered as money. The agents live only one period, and they are selfish do not care about future generations. So they inherit from their parents only shares $\theta_{j\,i}$ of domestic production and country territory.

1.1 Territory exchange mechanism

All agents live one time period. In the beginning of the period countries produce private good $f_i(S_i)$ then they can exchange their land for amounts ΔS_i, and produce with using changed territories in the end of the period $f_i(S_i + \Delta S_i)$. Citizens do not discount their consumption during the time period. All territories are used and their sum is constant $S_1 + S_2 = const$, hence the sum of territory changes ΔS_i is zero which means in our case of two countries

$$\Delta S_1 = -\Delta S_2 \tag{2}$$

Government of each country can transfer money (private good) personally to citizens of the other country at any moment in the time period. This is the key assumption allowing to compensate only to the selected part of the seller country society its loss of utility. Some of money transactions t_i to the country should be zero

$$t_1 + t_2 = 0, \tag{3}$$

unless there is no additional expenses, which will be considered later on. Group of country citizens that needs minimal compensation for a deal promotion is called the *ruling coalition* of the country. Government in each country maximizes the wealth of its ruling coalition.

1.2 Forms of governments and decision making

The exchange needs approval from both sides according to the national decision making systems. Due to the territory change ΔS_i production in the country changes at the following amount

$$\Delta f_i = f_i(S_i + \Delta S_i) - f_i(S_i). \tag{4}$$

In order person j of country i to agree with the deal her loss of utility $\theta_{j\,i}\Delta f_i(S_i)$ should be compensated. We will consider different formes of government implying different voting procedures.

- *Monarchy* is when only one person possesses the production and makes decisions which means absolute monarchy, i.e. $\theta_{N_i-1\,i} = 1$ and $\theta_{j\,i} = 0 \quad \text{for all } j = 1, \ldots, N_i - 1,$. Thus, in order to buy some land $-\Delta S_i$ from monarchy i one needs to compensate all loss of its domestic production Δf_i. The same total compensation is required in the case of unanimous voting rule when each citizen has veto power.

- *Oligarchy* is when citizens vote with their shares. Decision is made if people who vote for it own together more than one half of the domestic production. Here the buyer needs to compensate the half of the seller country production losses $\Delta f_i \frac{1}{2}$.

- *Democracy* has the rule of majority voting. In this case buyer needs to compensate the poorest half of the seller country i society with the total expenses $\Delta f_i \sum_{j=1}^{N_i/2} \theta_{j\,i}$.

Thus, the buyer needs to pay for the same amount of land its full cost to the Monarchy, half of its cost to Oligarchy, and less than half of full cost of the land to Democracy. That is because of the inequalities

$$\Delta f_i > \frac{1}{2}\Delta f_i \geq \Delta f_i \sum_{j=1}^{N_i/2} \theta_{j\,i}. \tag{5}$$

We introduce parameter $\alpha_i \in (0, 1]$ which reflects the decision mechanism in a country. When $\alpha_i = 1$ country i has Monarchy, when $\alpha_i = 1/2$ it is Oligarchy, and when $\alpha_i = \sum_{j=1}^{N_i/2} \theta_{j\,i}$ it is Democracy. Thus, coefficient α_i can also be a measure of inequality in country i if it is a Democracy.

Situation when country i buys land is different. Since shares $\theta_{j\,i}$ of future production remain constant all citizen with $\theta_{j\,i} > 0$ get benefit $\theta_{j\,i}\Delta f_i$ from the increase of country territory $\Delta S_i > 0$. Hence, all citizens agree to give their future $\theta_{j\,i}\Delta f_i$ for territory increase ΔS_i regardless of the government form in their country. Thus, each country i sell its territory for not less than $-\alpha_i \Delta f_i > 0$ and buy territory for not more than $\Delta f_i > 0$. It means that if $\alpha_i < 1$ then country i can sell territory cheaper than buy it.

2 Fare land trades

Let us consider fare land trading, where it is possible to pay for land after production and no conquest allowed. Here the production in the beginning of the period does not play a role and this product can be consumed, because country can pay for the territory increase after the production in the end of the period. Let us suppose that county 1 buys territory from country 2. Then we get the following maximization problem for the buyer

$$\begin{aligned} \Delta f_1 + t_1 &\rightarrow \max_{\Delta S_1 \geq 0,\, t_1 \leq 0}, && (6)\\ \text{s.t.} \quad \Delta f_1 + t_1 &\geq 0, && (7) \end{aligned}$$

where maximization of social welfare gain $\Delta f_1 + t_1$ is equivalent to the maximization of the ruling coalition welfare gain $\alpha_1 \Delta f_1 + \alpha_1 t_1$ because all citizens are ready to pay for increase of country territory since all of them benefit from it proportionally to their constant shares $\theta_{j\,1}$. In the seller country the not ruling part of the society is worse off because it does not get the money for their loss in future production share $(1 - \alpha_2)\Delta f_2$. Thus, the seller problem is as follows

$$\begin{aligned} \alpha_2 \Delta f_2 + t_2 &\rightarrow \max_{\Delta S_2 \leq 0,\, t_2 \geq 0}, && (8)\\ \text{s.t.} \quad \alpha_2 \Delta f_2 + t_2 &\geq 0. && (9) \end{aligned}$$

Expressions (6)–(9) form a bargaining problem. We assume that this problem can be solved somehow by the governments. We are interested only in the resulting territory allocation regardless the money transfers. We conclude that whatever the bargaining solution is both governments are interested in maximization of $\Delta f_1 + \alpha_2 \Delta f_2$

$$\begin{aligned} \Delta f_1 + \alpha_2 \Delta f_2 &\rightarrow \max_{\Delta S_1 = -\Delta S_2 \geq 0}, && (10)\\ \text{s.t.} \quad \Delta f_1 + \alpha_2 \Delta f_2 &\geq 0, && (11) \end{aligned}$$

where trade possibility condition (11) is obtained from the sum of budget constraints (7) and (9) taking into account relation (3).

We can define the price

$$p_i = \frac{\partial \Delta f_i}{\partial \Delta S_i} = \left.\frac{\partial f_i(S)}{\partial S}\right|_{S = S_i + \Delta S_i},$$

using expression (4). Thus, p_i is the maximal price at which country i can buy small part of territory. Then, $\alpha_i p_i$ is the price at which country i can sell small part of its territory.

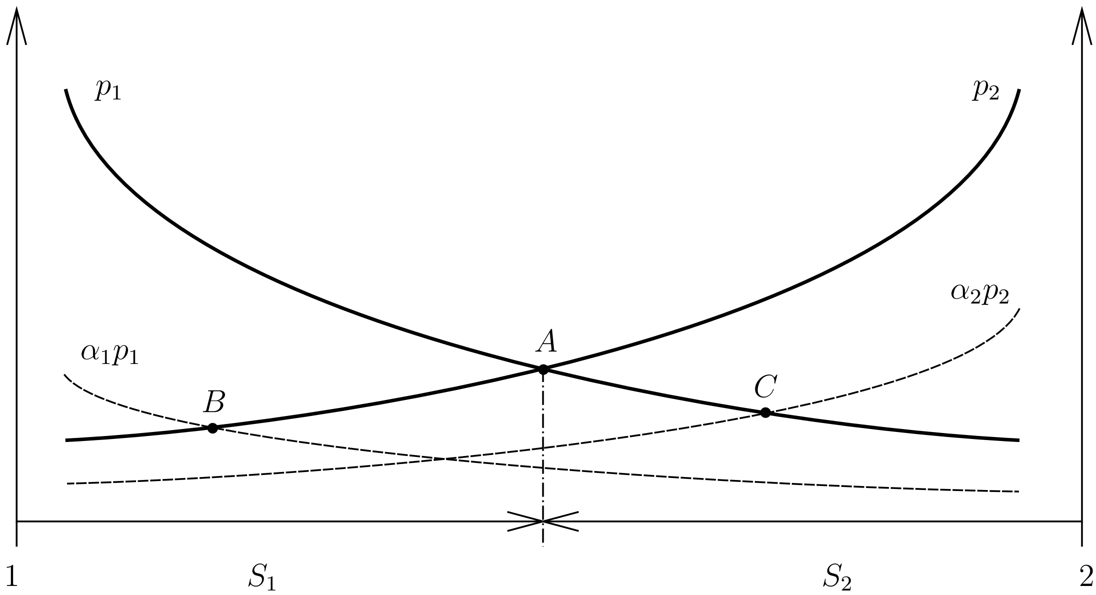

Figure 1: Functions p_1 and p_2 of territories S_1 and S_2 are the marginal territory productions of countries 1 and 2. Functions $\alpha_1 p_1$ and $\alpha_2 p_2$ are the marginal production shares of ruling coalitions in countries 1 and 2.

In Fig. 1 the prices p_i are drown with solid lines and the prices $\alpha_i p_i$ with dashed lines. In these terms the first order condition for problem (10) takes the following form

$$p_1 - \alpha_2 p_2 = 0 \quad \text{for } \Delta S_1 > 0, \tag{12}$$

$$p_1 - \alpha_2 p_2 \leq 0 \quad \text{for } \Delta S_1 = 0. \tag{13}$$

Condition (12) means that if initial border between countries is on the left from the absciss of point C in Figs. 1 and 2 then the countries will move the border to C. Condition (13) means that if the border is on the right of C country 1 will not buy any land.

In order to solve the same problem when country 2 buys territory from country 1 we need only to change indexes in (12) and (13)

$$p_2 - \alpha_1 p_1 = 0 \quad \text{for } \Delta S_2 > 0, \tag{14}$$

$$p_2 - \alpha_1 p_1 \leq 0 \quad \text{for } \Delta S_2 = 0. \tag{15}$$

Condition (14) means that if the initial border between countries is on the right from the absciss of point B in Figs. 1 and 2 then the countries will move the border to B. Condition (15) means that if the border is on the right of B country 2 will not buy any land.

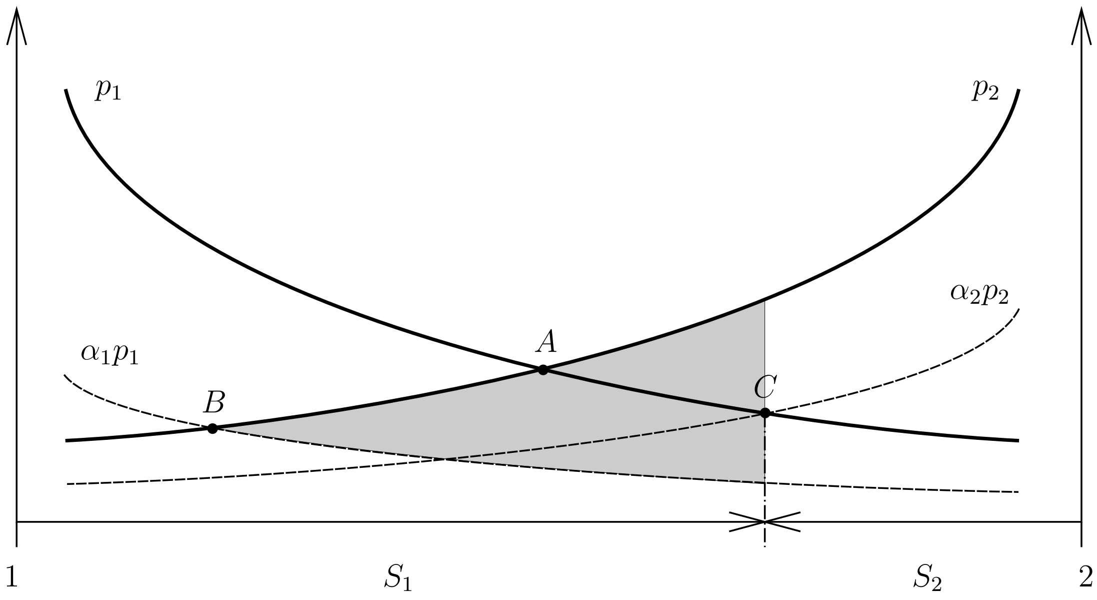

Figure 2: Country 1 buys territory from country 2 moving the border from point B to point C. Total trade revenue is the gray area $\Delta f_1^{BC} + \alpha_2 \Delta f_2^{BC}$.

From the solutions (12)–(15) we can conclude that if the initial border position is on the left from point B the only option is for country 2 to sell its lend to country 1 so that new border is at point C, and vice versa if the initial border position is on the right from point C the only option is for country 1 to sell its lend to country 2 so that new border is at point B. The only question left is at which point B or C countries will move their border if its initial position is between B and C. The answer, which country to be the buyer and which the seller, depends on the particular bargaining conditions. But we can assume as before that governments maximize the total revenue from the trade, then the border near or at point B will be moved to point C and vice versa the border near or at point C will be moved to point B, which creates a cycle in time. For example when country 1 is the buyer moving border from B to C the total revenue is $\Delta f_1^{BC} + \alpha_2 \Delta f_2^{BC}$ which is the gray area in Fig. 2, where we denote $\Delta f_i^{BC} = f_i(S_i^C) - f_i(S_i^B)$ according to (4). In the next period situation is opposite and border moves from C to B.

Thus we get a cyclical solution which is a sequence of bargaining results on the border position in isolated time periods. Generally speaking this solution is asymptotically stable in Lyapunov's sense (Liapounoff, 1907).

Definition 1 *Let $x(\tau)$ be the solution and $\tilde{x}(\tau)$ be the perturbed solution, where τ is* *the time. Solution $x(\tau)$ is (asymptotically) stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\tilde{x}(0) - x(0)| < \delta$ is sufficient for $|\tilde{x}(\tau) - x(\tau)| < \varepsilon$ for all $\tau > 0$ (and $\lim_{\tau\to\infty} \tilde{x}(\tau) = x(\tau)$).*

The only case when solution could be unstable is when the initial border position is at the threshold, where countries can not decide which country should sell or buy, because both options give the same total trade revenue. But such situation is not typical.

Let us introduce average size of the the country i

$$\langle S_i \rangle = \frac{S_i^B + S_i^C}{2}, \tag{16}$$

where S_i^B is the size of country when the border is at point B, while S_i^C is that at C. We can conclude from Figs. 1 and 2, that the smaller coefficient α_i of country i is the smaller its minimal size of territory is, while its maximal size of country i does not depend on α_i

$$\frac{dS_1^B}{d\alpha^1} > 0, \quad \frac{dS_2^C}{d\alpha^2} > 0, \quad \frac{dS_1^C}{d\alpha^1} = 0, \quad \frac{dS_2^B}{d\alpha^2} = 0. \tag{17}$$

Hence, the average size of country i is grater the grater parameter α_i is

$$\frac{d \langle S_i \rangle}{d\alpha_i} > 0. \tag{18}$$

It means that caeteris paribus the Monarchy in average is bigger than Oligarchy, which in tern bigger than Democracy. The greater inequality is in the Democracy, the smaller its territory could be.

2.1 Welfare analysis

Even without knowledge about exact values of money transfers t_i we can say something about the dependance between social welfare of the country and its type of government. Since agents have linear utilities the utilitarian social welfare function can be evaluated as production plus money transfer $W_i = f_i(S_i) + t_i$. Let us introduce the average social welfare of country i as

$$\langle W_i \rangle = \frac{f_i(S_i^B) + t_i^B + f_i(S_i^C) + t_i^C}{2}, \tag{19}$$

where S_i^B and t_i^B are the territory and money transfer of country i if the border is at point B; while S_i^C and t_i^C are those at C in Fig. 2. Thus, regardless of the governments' bargaining powers we can write the lower bounds of countries' welfares

$$\langle W_1 \rangle \geq f_1(S_1^B), \quad \langle W_2 \rangle \geq f_2(S_2^C). \tag{20}$$

The global average social welfare can be calculated precisely from (19) because of the payment balance (3)

$$\langle W_1 + W_2 \rangle = \sum_{i=1}^{2} \frac{f_i(S_i^B) + f_i(S_i^C)}{2} \tag{21}$$

If we differentiate expression (21) by parameters α_i, then using conditions (1) and (17), then we get that the global average social welfare increase when parameters α_i increase

$$\frac{d \langle W_1 + W_2 \rangle}{d\alpha_i} > 0. \tag{22}$$

Thus, we can get the unique global social optimum for the corner solution $\alpha_i = 1$, which implies that countries are Monarchies and locate the border at point A.

2.2 Two Monarchies choose the stable social optimum

When two Monarchies trade territory the buyer have to compensate all domestic production loss to the seller since $\alpha_1 = \alpha_2 = 1$. That is why maximization problems (6)–(7) and (8)–(9) coincide

$$\Delta f_1 + \Delta f_2 \quad \to \quad \max_{\Delta S_1 = -\Delta S_2}, \tag{23}$$

$$\text{s.t.} \quad \Delta f_1 + \Delta f_2 \quad \geq \quad 0, \tag{24}$$

and have the same unique solution $p_1 = p_2$ positioning the border at point A in Fig. 1.

Proposition 1 *If optimal $\Delta S_i \neq 0$ in (23) then constraint (24) is strict inequality.*

This territory allocation is a global social optimum as it actually maximizes the sum of production in both countries

$$\Delta S_1^A = \arg \max_{\Delta S_1 = -\Delta S_2} f_1(S_1 + \Delta S_1) + f_2(S_2 + \Delta S_2) = \arg \max_{\Delta S_1 = -\Delta S_2} \Delta f_1 + \Delta f_2.$$

That is why this allocation is Pareto efficient. We note that the territory allocation of two Monarchies is a steady state equilibrium which means that if time period starts with such allocation the Monarchies will not change it. This steady state solution is also asymptotically stable in Lyapunov's sense; see definition 1, which means that if period starts with slightly different allocation then it converges back to that at point A.

3 Effect of possibility to cheat and conquest

Let us add to the model the possibilities of not executing a contract (cheating) and conquest of the territory. The conquest is a two step sequential game, see Fig. 3. In the first step country 1 (conqueror) instead of paying for the territory, pays $-t_1^a \geq 0$ for hiring the army to conquer a part of neighbor's territory. In the second step if the country under attack does not pay equal or greater amount of money $-t_2^a$ to the army to defend its territory then the territory goes to the conqueror.

We recall that private good is produced twice in the time period in the beginning and in the end. We assume that between these two productions countries play the game in Fig. 4 which includes two conquest supgames, see Fig. 3. Thus, before the game each country is endowed with the good from the first production. Only that amount of good can be used for land purchasing, because no one would rely on the promise to pay in the end of the period if cheating is allowed. Hence, there is the initial money constraint

$$f_i(S_i) + t_i^A \geq 0, \tag{25}$$

which matters only for the conqueror as will be shown later on.

Conquest looks like country 1 buys lang from Monarchy (see Fig. 1) having budget constraint (25). The only difference is that the seller does not get a payment

$$-t_1^a = \min\left\{-\Delta f_2^A, f_1(S_1)\right\} \tag{26}$$

which goes to the Army. The Army could be treated as the third player which can not gain utility from the land itself but only conquers territory for the country which pays the highest price.

In the conquest game in Fig. 3 country 2 will wish to defend its territory ΔS_2 with minimal military expanses $-t_2^a = -t_1^a$ if conqueror pays $-t_1^a < -\Delta f_2$, otherwise country 2 retreats with $t_2^a = 0$.

Proposition 2 *Initial production $f_i(S_i)$ of country i is grater than loss $-\Delta f_i > 0$ from any territory decrease $\Delta S_i \in [-S_i, 0)$, i.e. $f_i(S_i) > -\Delta f_i$.*

It follows from definition (4) of symbol Δf_i and first property of function $f_i(S_i)$ in (1).

Corollary 1 *In order to conquer the territory ΔS_1 country 1 should transfer $t_1^a = \Delta f_2$*

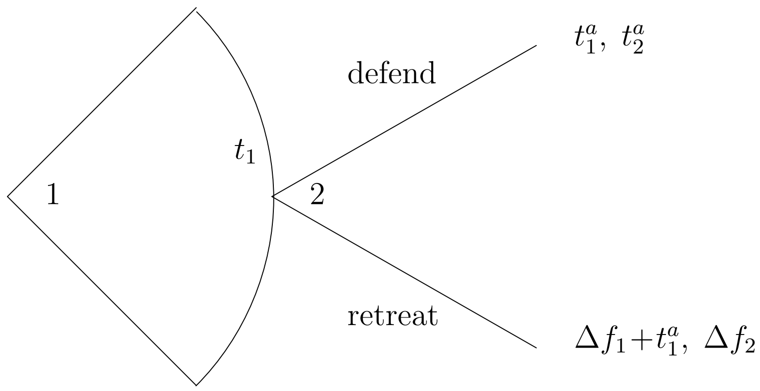

Figure 3: Conquest game. Payoffs of countries are sums of payoffs of all its citizens. Conqueror chooses military transfer $t_1^a < 0$. Attacked country defends spending $t_2^a = t_1^a < 0$ if $t_1 > \Delta f_2$, or retreats spending $t_2^a = 0$ if $t_1^a \leq \Delta f_2$. That is why in SPNE conqueror chooses $t_1^a = \Delta f_2$.

It follows from proposition 2 that country 2 has enough initial endowment $f_2(S_2)$ to defend its territory which it wishes if $-t_1^a < -\Delta f_2$. Hence, inequality (25) is strict and $-t_1^A = -\Delta f_2$ is the minimal payment for conquest. ■

Using corollary 2 we can write the maximization problem of the conqueror in the game Fig. 3

$$\Delta f_1 + \Delta f_2 \quad \to \max_{\Delta S_1 = -\Delta S_2 \geq 0}, \tag{27}$$

$$\text{s.t.} \quad f_1(S_1) + \Delta f_2 \quad \geq \quad 0, \tag{28}$$

which differs from (23)–(24) only by constraint (28). The Supgame Perfect Nesh Equilibrium (SPNE) of the conquest game in Fig. 3 consists of the payments to the army (26), $t_2^a = 0$, and territory changes ΔS_i which can be found in both cases when the following inequality is satisfied and not

$$f_1(S_1) > -\Delta f_2^A. \tag{29}$$

1. If inequality (29) is satisfied we will say that country 1 is *rich* then it conquers ΔS_1^A. Conqueror's profit is $\Delta f_1^A + \Delta f_2^A \geq 0$ while the profit of the seller's country $\Delta f_2^A \leq 0$ and profit of its ruling coalition is $\alpha_2 \Delta f_2^A \leq 0$.

2. If inequality (29) is not satisfied, which means that country 1 is *not rich*, then it conquers such amount of territory $\Delta S_1^a \leq \Delta S_1^A$ that $f_1(S_1) = -\Delta f_2$.

We will analyze the game in Fig. 4 for both internal solution for rich conqueror and corner solution for not rich one.

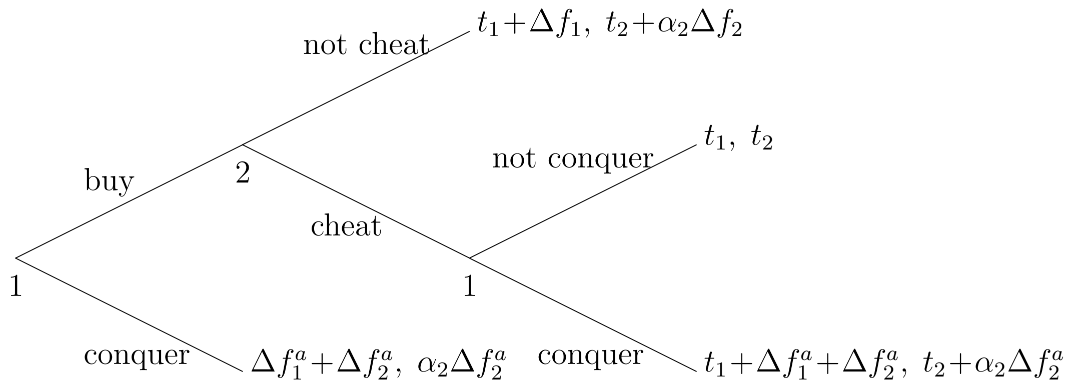

Figure 4: The game territory exchange is played each time period by new generation. Payoffs of country 1 are sums of payoffs of all its citizens. Payoffs of country 2 are sums of payoffs of its ruling coalition members.

If country 1 decides to buy land ΔS_1 then the government of country 1 proposes to its citizens a project and collect money $-t_1$ from those of them who will benefit from the territory increase. It means that all who have strictly positive share $\theta_{j\,1}$ of production would agree to pay not more than $\theta_{j\,1}\Delta f_1$ and hence would vote for the project. Then the government of 1 gives money $t_2 = -t_1 \leq \min\{-f_1(S_1), \Delta f_1\}$ to the ruling coalition of country 2.

The ruling coalition of country 2 receives the money t_2 and can execute its duty giving required territory $-\Delta S_2$ to the buyer or can cheat refusing to give the territory. In the first case the profit of the seller's ruling coalition is $t_2 + \alpha_2\Delta f_2$, profit of the buyer is $t_1 + \Delta f_1$, where transfers are balanced according to (3), and the game is finished. In the case of cheating the profit of seller is t_2, while the profit of buyer is $t_1 < 0$. But the buyer has the possibility to conquer part ΔS_1^a of the disputed territory paying for this Δf_2^a as much as the seller maximally can pay to keep that territory. In that case being rational the seller country retreats without fight and its ruling coalition have profit $t_2 + \alpha_2\Delta f_2^a$ while the profit of the buyer is $t_1 + \Delta f_1^a + \Delta f_2^a$, and the game is finished.

Proposition 3 *Country* 1 *buys only such amount of land* ΔS_1 *which it will be able to conquer in case of cheating by county* 2.

Country 1 buys land when it surely is not going to be cheated, because otherwise it would conquer it at first. If county 1 byes more than it can conquer being cheated it gives country 2 incentives to cheat, because country 2 will have more land rest after conquest than after executing its duties. ■

Corollary 2 *Country* 1 *buys land of amount* ΔS_1 *not exceeding the amount* ΔS_1^A *that locate the border at point A, like it would trade with Monarchy.*

Since point A is the optimal position of the border for unconstrained conquest (27) country 1 would never conquer more than ΔS_1^A after being cheated. It follows from proposition 3 that country 1 also do not buy more then ΔS_1^A. ■

It follows from corollary 2, that maximization problem for buying the lend

$$\Delta f_1 + t_1 \quad \rightarrow \quad \max_{\Delta S_1 \geq 0,\, t_1 \leq 0}, \tag{30}$$

$$\text{s.t.} \quad f_1(S_1) + \Delta f_2 + t_1 \quad \geq \quad 0, \tag{31}$$

has more restrictive budget constraint (31) than (7). If both countries prefer to trade the bargaining problem is composed by (30)–(31) and (8)–(9) with conquest (27)–(28) as a disagreement point.

1. When country 1 is *not rich* (inequality (29) is not satisfied) then inequality (31) becomes an equality. Hence, ΔS_1^a is also solution of maximization problem (30)–(31), which yields profit $\Delta f_1^a + \Delta f_2^a + t_1$, where $t_1 \leq 0$. Country 1 chooses between conquest and purchase comparing their profits $\Delta f_1^a + \Delta f_2^a$ and $\Delta f_1^a + \Delta f_2^a + t_1$, therefore, it would buy only for free, i.e. $t_1 = 0$, which is the disagreement point for seller country 2. Thus, not rich country conquers territory rather then buys it.

2. When country 1 is *rich* (inequality (29) is satisfied) it buys ΔS_1^A according to corollary 2, with payment $t_2 = -t_1 \leq f_1(S_1) + \Delta f_2^A$.

These strategies are SPNE of the game in Fig. 4 which we can match to those in (Grossman & Mendoza, 2001, 2004). We can distinguish the type of annexation according to the values of t_2.

- We can say that if $t_2 + \alpha_2 \Delta f_2^A \geq 0$ then it is an Uncoerced Annexation, because the ruling coalition of country 2 is better off in the end of the period.
- If $t_2 + \alpha_2 \Delta f_2^A < 0$ while $t_2 > 0$ then it is a Coerced Annexation, because the ruling coalition of country 2 is worse off after the trade. Which means that country 1 threatened it by possibility of conquest. Though, we have not any additional costs introduced for Coerced Annexation, like the cost of deploying Legions in (Grossman & Mendoza, 2001, 2004).
- If the countries have nothing to bargain because country 1 is not rich, then it simply conquers the territory ΔS_1^a, which we would call an Attempted Conquest, where $t_2 = 0$.

Thus, not rich country conquers territory generation by generation until it becomes rich enough to buy territory till the globally optimal size S_i^A.

The main result of this section is that the possibility to cheat and conquest completely removes the dependence of the country size on its type of government. More then that, this possibility results in the Pareto efficient and asymptotically stable steady state territory allocation like that between two Monarchies with border at point A in Fig. 1.

4 Effect of additional costs

Until now we believed that the there is no costs for adaptation of new territory or moving the border. Let us assume that there are adaptation cost $d \geq 0$ per unit of traded territory and constant cost $c \geq 0$ of rebuilding the border, then money balance equation (3) takes the following form

$$t_1 + t_2 = -c - d\,|\Delta S_1|\,. \tag{32}$$

Hence, maximization problem (10)–(11) also changes

$$\Delta f_1 + \alpha_2 \Delta f_2 - c - d\,|\Delta S_1| \;\; \rightarrow \max_{\Delta S_1 = -\Delta S_2 \geq 0}, \tag{33}$$

$$\text{s.t.} \quad \Delta f_1 + \alpha_2 \Delta f_2 - c - d\,|\Delta S_1| \;\; \geq \;\; 0, \tag{34}$$

The first order condition for problem (33) takes the following form

$$p_1 - \alpha_2 p_2 - d \;\; = \;\; 0 \quad \text{for } \Delta S_1 > 0, \tag{35}$$

$$p_1 - \alpha_2 p_2 \;\; \leq \;\; 0 \quad \text{for } \Delta S_1 = 0. \tag{36}$$

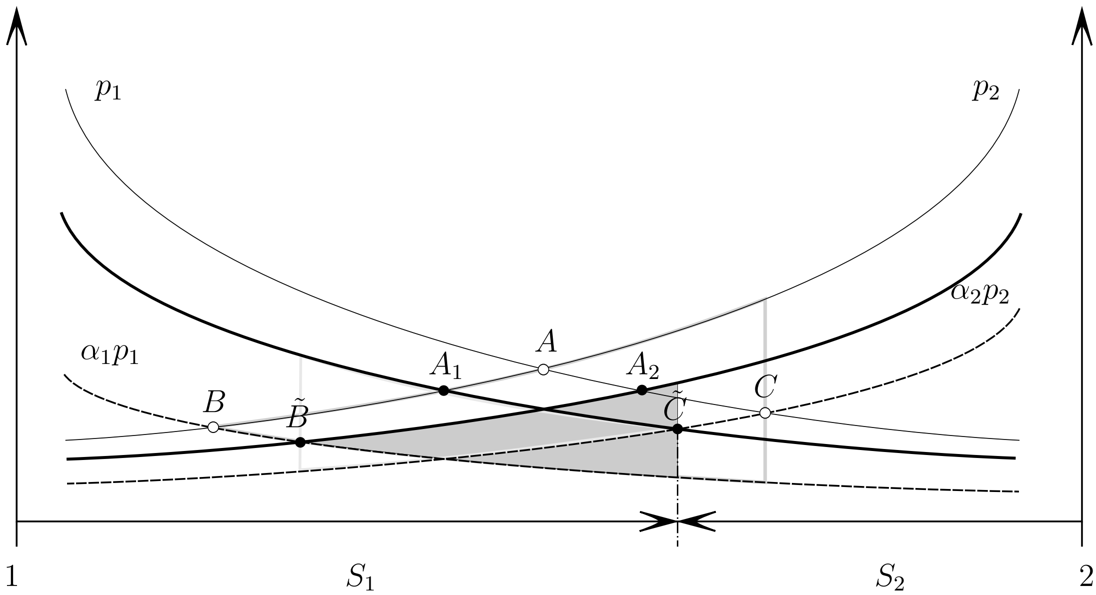

Figure 5: Country 1 buys territory from country 2 moving the border from point B to point C. Total trade revenue $\Delta f_1^{\tilde{B}\tilde{C}} + \alpha_2 \Delta f_2^{\tilde{B}\tilde{C}} - c - d \left| \Delta S_1^{\tilde{B}\tilde{C}} \right|$ is the gray area minus constant c.

We see that only per unit adaptation cost d influences the first order condition (35)–(36), and hence, moves points B and C to $\tilde{B}$ and $\tilde{C}$, see Fig. (5), while constant c in the maximization constraint (11) determines whether trade will happen or not. Thus, oscillatory solutions can occur only if both of the following inequalities are satisfied

$$\Delta f_1^{\tilde{B}\tilde{C}} + \alpha_2 \Delta f_2^{\tilde{B}\tilde{C}} - c - d \left(S_1^{\tilde{C}} - S_1^{\tilde{B}} \right) \geq 0, \tag{37}$$
$$\Delta f_2^{\tilde{C}\tilde{B}} + \alpha_1 \Delta f_1^{\tilde{C}\tilde{B}} - c - d \left(S_2^{\tilde{B}} - S_2^{\tilde{C}} \right) \geq 0. \tag{38}$$

It is easily seen form Def. (1), that if inequalities (37) and (38) are strict for all points of a solution then that oscillatory solution is asymptotically stable. Oscillatory solutions could coexist with multiple steady state solutions, these are all such points S_i for which the following inequalities are satisfied

$$\Delta f_1^{\tilde{C}} + \alpha_2 \Delta f_2^{\tilde{C}} - c - d \left(S_1^{\tilde{C}} - S_1 \right) < 0, \tag{39}$$
$$\Delta f_2^{\tilde{B}} + \alpha_1 \Delta f_1^{\tilde{B}} - c - d \left(S_2^{\tilde{B}} - S_2 \right) < 0. \tag{40}$$

These steady state solutions are stable but not asymptotically. Here it means that if we change the initial point a little (so that inequalities (39) and (40) are still satisfied), then solution neither goes further away from the unperturbed solution, nor converges back to it.

If cheating and conquest are allowed then the same game theoretic reasoning as in the previous section gives us the existence of steady state points S_i, which should satisfy the following inequality

$$\Delta f_1^{\tilde{A}} + \Delta f_2^{\tilde{A}} - c - d \left| S_i^{\tilde{A}} - S_i \right| \quad < \quad 0. \tag{41}$$

It means that all points between A_1 and A_2 are stable steady state solutions.

We can conclude that additional linear costs reduce the amplitude of territory oscillations. Hence, they decrease dependence between size of the country and its type of government. Such costs can also cause the appearance of multiple stable steady state positions of the border.

Conclusion

In this work the mechanism governing the territory changing between groups of people such as countries is proposed, based on trading with approval of both sides under particular voting rule (veto rule, majority rule,...).

We got a cyclical in time asymptotically stable solutions which is sequence of bargaining results on the border position. We also found that the average size and social welfare of the Monarchy is bigger than those of Oligarchy, which in tern bigger than those of Democracy. The greater inequality is in the Democracy, the smaller its territory could be.

Possibility to cheat and conquest completely removes the dependence of the country size on its type of government. More then that, this possibility results in the Pareto efficient and asymptotically stable steady state territory allocation like that between two Monarchies. It was found that only rich countries can afford itself to buy territory rather then conquer it.

Additional linear costs reduce the amplitude of territory oscillations. Hence, they decrease dependence between size of the country and its type of government. Such costs can also cause the appearance of multiple stable steady state positions of the border.

Appendix

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