Efficient and Strategy-Proof Voting over Connected Coalitions

Veronica Block*

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1 Motivation

By the Gibbard-Satterthwaite-Theorem ([2] and [6]) the only strategy-proof voting rule on an unrestricted preference domain is the dictatorship of one individual if there exist at least three alternatives. For possibility results, restrictions of the domain are necessary. Important examples are the domain of all single-peaked preferences on a line (see Moulin [3]) and the domain of all separable preferences on the hypercube (see Barberá, Sonnenschein, Zhou [1]).

In this paper a novel example of a possibility domain is presented. As the two preference domains mentioned above (and a number of other possibility domains as well), it belongs to the class of generalized single-peaked domains developed by Nehring and Puppe [5].

To motivate our preference domain, consider a finite set of political parties ordered from left to right on the political spectrum. The space of alternatives is the family of all *connected* coalitions, i.e. the family of all non-empty coalitions that contain with any two parties all parties that are between them in the political spectrum. The family of connected coalitions can be endowed with a natural betweenness relation as follows: a connected coalition C is *between* two connected coalitions C_1 and C_2 if (i) the leftmost element of C is between the two leftmost elements of C_1 and C_2, and (ii) the rightmost element of C is between the two rightmost elements of C_1 and C_2.

*University of Karlsruhe, e-Mail: Veronica.Block@kit.edu

A preference ordering on the family of all non-empty connected coalitions is called *generalized single-peaked* if it admits a unique peak, say C^*, such that a coalition C is strictly preferred to another coalition C' whenever C lies between C' and C^*.
Using the general characterizations of [5], we show that on the domain of all generalized single-peaked preferences over connected coalitions there exists an anonymous, neutral and strategy-proof voting rule. It selects the connected coalition which has as leftmost element the median of the leftmost elements of the individually most preferred coalitions and as rightmost element the median of the rightmost elements of the individually most preferred coalitions. The key for the application of the results of [5] is the observation that the space of all connected coalitions endowed with the above betweenness relation forms a *median space*. Moreover, this median space is two-dimensional. This implies by the analysis of [4] that the above voting rule is *efficient*.

2 The model

Let $A = \{a_1, \dots, a_m\}$ be a finite set containing $m \geq 2$ objects. We consider the case in which individuals have preferences over the power set $\mathcal{P}(A)$ consisting of all subsets of A.
By the Gibbard-Satterthwaite-Theorem all strategy-proof social choice functions on an unrestricted preference domain over $\mathcal{P}(A)$ are dictatorial.

We consider the following domain restriction: Let $<$ be a linear ordering of A, w.l.o.g. $a_1 < \dots < a_m$. As a specific example one may think of A as representing a set of political parties which can be ordered from left to right on a political spectrum. In this case the power set $\mathcal{P}(A)$ represents the class of possible coalitions. While other interpretations may be applicable as well, in the remainder will refer to the elements of A as political parties and to the elements of $\mathcal{P}(A)$ as coalitions. For notational convenience we identify parties with their indices and simply write (ijk) for $\{a_i, a_j, a_k\}$. We denote by $\mathcal{C}_< \subset \mathcal{P}(A)$ the set of all connected coalitions[1]:

[1]Remark: The cardinality of $\mathcal{C}_<$ grows quadratically in m, i.e. $\#\mathcal{C}_< = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$.

A non-empty coalition $C \in \mathcal{C}_<$ is called **connected**, if

$$\forall i, j \in C \text{ and } i < k < j \Rightarrow k \in C.$$

For every connected coalition C we call

$$l_C \in C \text{ **leftmost in C** if for all } k < l_C \Rightarrow k \notin C$$

and

$$r_C \in C \text{ **rightmost in C** if for all } k > r_C \Rightarrow k \notin C.$$

Evidently one has $C = (l_C ... r_C)$ for every connected coalition C.

Example: The coalition $C = (234)$ consisting of the parties a_2, a_3 and a_4 is connected with $l_C = 2$ and $r_C = 4$. $C' = (24)$ is not connected and therefore no element of $\mathcal{C}_<$.

We define the following betweenness relation on $\mathcal{C}_<$. A coalition C is **between** C_1 and C_2 if

- its leftmost party l_C is between l_{C_1} and l_{C_2} and
- its rightmost party r_C is between r_{C_1} and r_{C_2}.

Formally, C is between C_1 and C_2 if

$$l_C \in [\min\{l_{C_1}, l_{C_2}\}, \max\{l_{C_1}, l_{C_2}\}] \text{ and } r_C \in [\min\{r_{C_1}, r_{C_2}\}, \max\{r_{C_1}, r_{C_2}\}].$$

For graphical illustration of the betweenness relation consider Figure 1 with $m = 6$ parties. A coalition C is between C_1 and C_2 if and only if it lies on a shortest path connecting C_1 and C_2 on the graph. Note that shortest paths need not be unique.

Obviously the betweenness relation respects the subset ordering, i.e. a coalition is between its super- and subsets. (see Figure 2) The neighbours of a coalition $(l_C \dots r_C)$ are all coalitions which are connected and which consist either of exactly one party more $((l_C - 1 \dots r_C), (l_c \dots r_C + 1))$ or one party less $((l_C \dots r_C - 1), (l_C + 1 \dots r_C))$.
We are now able to define the preference structure over connected coalitions.

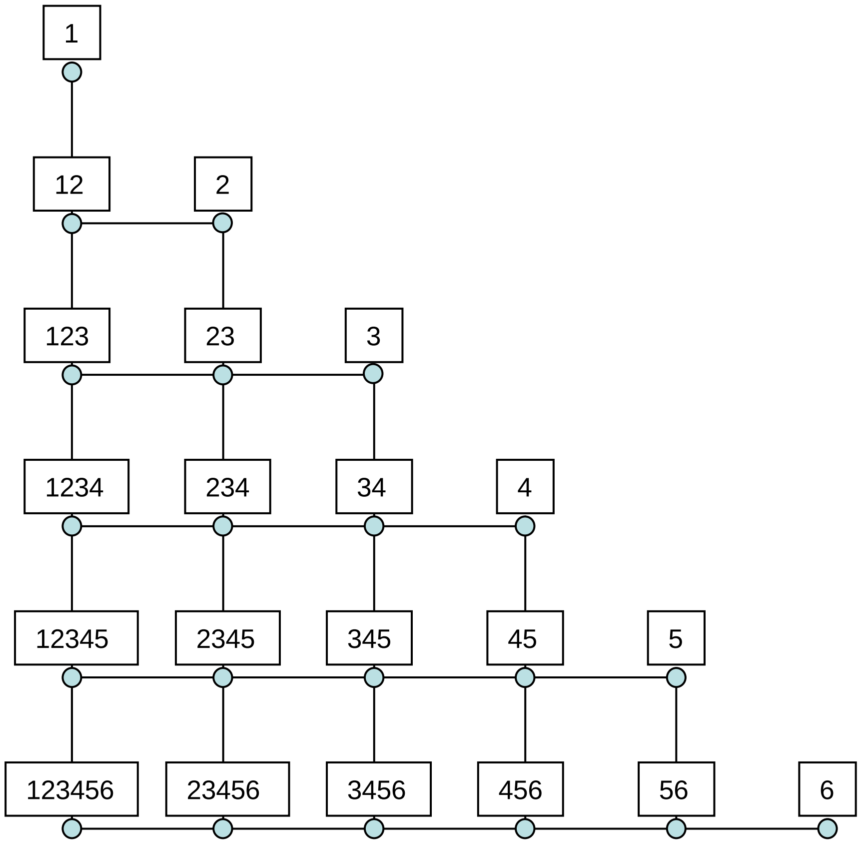

Figure 1: Graphical illustration for six parties

Let $N = \{1, \ldots, n\}$ denote the set of voters[2]. Suppose that every individual i has a unique favourite coalition $C_i^* = (l_{C_i}^* \ldots r_{C_i}^*)$ which is called **peak of i**, i.e. for all $C \in \mathcal{C}_<$

$$C \neq C_i^* \Rightarrow C_i^* \succ_i C.$$

The preference relation $(\succsim_i)$ of individual i is **generalized single-peaked** if for all connected coalitions C and $C' \neq C_i^*$ the condition holds:

$$C \text{ is between } C_i^* \text{ and } C' \Rightarrow C \succ_i C'$$

Denote by $\mathcal{S}(\mathcal{C}_<)$ the set of all generalized single-peaked preferences on $\mathcal{C}_<$.

Example: Suppose that $\succ$ is generalized single-peaked with peak $C_i^* =$ (234). Then, for instance, $(234) \succ_i (23) \succ_i (123)$, but there is no restriction

[2]For simplicity we assume throughout that the number of voters is odd.

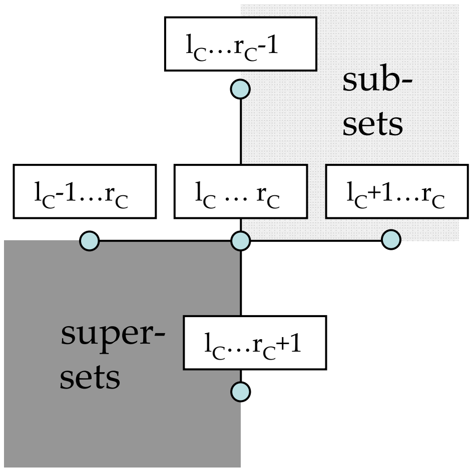

Figure 2: The coalition C with its neighbours

on the preference over (23) and (1234).

Remark: The concept of generalized single-peakedness cannot be reduced to single-peakedness in the classical sense, i.e. with respect to a linear ordering.
(Proof: Let $i, j, k \in N$ be three individuals with the same peak C^*. Suppose there exists a linear-ordering, such that the preferences are single-peaked. Then each alternative which is on the second place in the preference ordering of an individual must be a neighbour of C^* in the linear ordering. As a coalition may have more than two neighbours, this leads to a contradiction. $\lightning$)

For a profile $(\succsim_1, \ldots, \succsim_n)$ of n preference relations, the tuple consisting of the peaks $\mathbb{C}^* := \{C_1^*, \ldots, C_n^*\}$ is called **peak-profile**.
A social choice function is a mapping

$$F := \begin{cases} \mathcal{S}(\mathcal{C}_<)^n & \longrightarrow \quad \mathcal{C}_< \\ (\succsim_1, \ldots, \succsim_n) & \longmapsto \quad C \end{cases}$$

F is called **strategy-proof** if for all $i \in N$ and $\succsim_i, \succsim_i' \in \mathcal{S}(\mathcal{C}_<)$:

$$F(\succsim_1, \ldots, \succsim_i, \ldots, \succsim_n) \succsim_i F(\succsim_1, \ldots, \succsim_i', \ldots, \succsim_n)$$

3 Results

Consider the social choice function

$$\widehat{F}(\succsim_1, \ldots, \succsim_n) = (med\{l_{C_1^*}, \ldots, l_{C_n^*}\}, \ldots, med\{r_{C_1^*}, \ldots, r_{C_n^*}\})$$

where med denotes the median-operator, i.e. $medX$ satisfies
$\#\{x \in X : x \leq med\} = \#\{x \in X : x \geq med\}$.

Theorem 1.
$\widehat{F}(\cdot)$ is a strategy-proof, anonymous, neutral.

Proof: For the proof we can use results from Nehring and Puppe [5]. For this it is sufficient to verify that the space of connected coalitions endowed with our notion of betweenness is a median space. □

Example: There are $m = 6$ parties and $n = 5$ individuals with peaks on the coalitions $C_1^* = (2), C_2^* = (123), C_3^* = (34), C_4^* = (45), C_5^* = (23456)$ respectively. The median of the leftmost parties is $med\{2, 1, 3, 4, 2\} = 2$ and the median of the rightmost parties is $med\{2, 3, 4, 5, 6\} = 4$. Therefore $\widehat{F}(\succsim_1, \ldots, \succsim_n) = (234)$.

Proposition 2.
$\widehat{F}(\cdot)$ is efficient, i.e. for all $(\succsim_1, \ldots, \succsim_n)$ there exists no $C \in \mathcal{C}_<$ such that $C \succeq_i \widehat{F}(\succsim_1, \ldots, \succsim_n)$ with at least one strict preference.

Proof: Since the associated graph (see Figure 1 can be embedded in a two-dimensional space, we can apply the main theorem of Nehring and Puppe [4], which shows that $\widehat{F}(\cdot)$ is efficient. □

4 Conclusion

... to be written ...

References

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