Charitable Asymmetric Bidders*

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Abstract

Recent papers show that all-pay auction is better at raising money for charity than first-price auction with symmetric bidders and under incomplete information. Yet, this result is lost with bidders enough asymmetric and under complete information. In this paper, we consider a framework on charity auctions with asymmetric bidders under some incomplete information. We determine all-pay auction still runs more money than first-price auction. Thus, all-pay auctions should be seriously considered when one wants to organize a charity auction.

KEYWORDS: All-pay auctions, Charity, Externalities
JEL CLASSIFICATION: D44, D62

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1 Introduction

Fundraising activities for charitable purposes have become increasingly popular. One reason is the growing number of non-governmental organization with humanitarian or social purposes. Another one is the decrease of government participation in culture, education and related activities. The purpose of these associations are either the development and promotion of culture or aid and humanitarian services. Even in France, a country without any fundraising tradition, some organizations began to appear, such as the *French Association of Fundraiser*[1] in 2007.

Commonly used mechanisms to raise money are voluntary contributions, lotteries and auctions. Even if most of the fundraisers use voluntary contributions[2], auctions are increasingly used. Indeed, for some special events or particular situations, auctions provides a particular athmosphere. The popularity of auctions for charity purposes can also be observed by the increase in internet sites offering the sale of objects and donating a part of their proceeds to charity. Well-known examples include *Yahoo!* and *Giving Works* of *eBay*. Many others have been created, such as the *Pass It On Celebrity Charity Auction*[3] in 2003, where celebrities donated objects whose sale revenue contributed to a "charity of the month". We can also cite *cMarket Charitable Auctions Online*[4] created in 2002 and selected as a charity vehicle by more than 930 organizations.

Consequently, there is a growing and recent literature on charity auctions. Goeree et al. (2005) and Engers and McManus (2007) investigate an independent private values model and show that all-pay auctions are better at raising money for charity than winner-pay auctions. Moreover, Onderstal and Schram (2008) lead a lab experiment and confirm these theoretical results. However, Carpenter et al. (2008) run a field experiment in four american preschools. In their experiments the ranking of the revenues is reversed. They attribute this result to the unfamiliarity of the participants to the mechanism and endogenous participation (see Carpenter et al. (2007) for a theoretical justification of the endogenous participation). However, we can also investigate this question in a situation where people are different in the sense that they do not have the same believes. Indeed, the theory assumes that people have the same altruism parameter and valuations are drawn from the same distribution. Bos (2008) provides an answer with complete information. He investigates a model with complete information and heterogeneity on the bidders' values. Then, he shows that when the asymmetry among bidders is strong enough, the ranking of revenues is reversed. In particular, the winner-pay auctions outperfom all-pay auctions.

[1] http://www.fundraisers.fr/

[2] There is further evidence of this phenomenon on the Internet with the emergence of sites such as http://www.JustGive.org.

[3] http://www.passitonline.org/

[4] http://www.cmarket.com/

Then the point of this paper is then to determine, whether all-pay auctions are still better at raising money for charity when bidders are asymmetric under some incomplete information. This result is not obvious as Bos (2008) and Carpenter et al. (2007) showed recently that it could be lost either with asymmetric bidders under complete information or with endogenous participation with symmetric bidders under incomplete information. Moreover, if we conclude that all-pay auctions are still better with asymmetric bidders and incomplete information we should consider implementing all-pay auctions to raise money for charity in some environments. Indeed, to the best of our knowledge, all-pay auctions have never been implemented in real life for charity purposes. However, it seems easy to do it. For example, every bidder could buy a number of tickets simultaneously as in a tombola. Contrary to a tombola, though, the winner will be the buyer with the highest number of tickets in hand.

In charity auctions, bidders make their bid decisions taking into account two parameters. Their valuation for the item sold and their altruism or sensitivity to the charity purpose. Asymmetry in the former or the latter parameters leads to the same qualitative results. In this paper we consider valuations drawn with the same distribution in an independent private values model. Then, we introduce asymmetry in the altruism parameter with complete information. As in Bulow et al. (1999), this framework presents the advantage to avoid the complexity and the narrow results of asymmetric auctions with incomplete information. Indeed, in the usual asymmetric auction literature, valuations are drawn from different distributions. Then changing these distributions could change the ranking of the revenue among different auction designs (for example, see Krishna (2002)). Yet, Maskin (2000), de Frutos (2000) and Cantillon (2008) succeed in determining the revenue ranking between first-price and second-price auctions under some conditions that the distributions should satisfy. Consequently, in this literature, distributions of the bidders' value are crucial elements.

This paper is closest the spirit to Bulow et al. (1999). They compare first-price and second-price auctions in independent private signals model with common values and two bidders. The signals are drawn from the uniform distribution and some parameters, that could be interpreted as altruism parameter to the charity purpose, are asymmetric under complete information. Although they apply this framework to toeholds and takeovers, it is well suited for charity. In their paper, they determine that when these parameters are asymmetric and small enough, the revenue ranking could be reverse such that the first-price outperforms the second-price auction. Contrary to them, we compare first-price to all-pay auctions in an independent private values model. The only one other paper on asymmetric auctions with this kind of externalities is de Frutos (2000). She compares first-price and second-price auctions with altruism parameters equal to $1/2$ and bidders' values drawn from different distribution. Then, her framework is quite different to ours as she does not investigate all-pay auctions and the asymmetry concerns bidders values and not altruism parameters. However, dividing our all-pay auction by 1 minus the bidder's altruism parameter leads to study the all-pay auction in her framework with uniform distributions.[5] Thus, in a technical way, our papers are connected.

Section 2 sets out our simple model of two bidders with altruistic asymmetric parameters that have independent private values about the item sold. Then in Section 3 and 4 we characterize the bidding equilibrium strategies for the all-pay auction and the first-price auction. In Section 5, we compare revenues and show that all-pay auction still outperforms first-price auction independently of level of asymmetry in their sensibility parameter. All the proofs are in the Appendix.

2 Preliminaries

Suppose two bidders take part in an auction through a fundraising event such as a charity dinner. Each bidder is risk neutral and cares about how much the other bidder pays in the auction. Indeed, as the amount of money will be used for a charity purpose, the bidders include in their utility function the bid of their competitor. Thus, their bidding functions depend of two parameters. Their valuation of the object sold and their altruism or their interest for the charity purpose that the auction should finance. The more a bidder is sensitive to the charity event the higher this parameter will be. Denote as v_i the valuation and as a_i bidder i's altruism parameter. Bidder valuations V_1, V_2 are independently and identically distributed and we normalize them to uniform distributions on $[0, 1]$. Moreover, the altruism parameters are common knowledge and heterogeneous such that $a_1 > a_2$. Then, bidder 1 has a higher preference for the charity purpose than bidder 2. When a bidder takes part in a charity auction, she obtains a positive externality from the amount of money raised. Indeed, she hopes that the highest amount will be collected to finance the charity event. Then, she would benefit from a percentage of the revenue collected as a return from the bids paid. In this paper we consider two auction designs: the first-price all-pay auction, usually called all-pay auction, and the first-price winner-pay auction which is the usual first-price auction.

In the all-pay auction the winner as well as the losers pay their own bid. Yet, each bidder receives an externality from her own bid as well as from her competitor's. Denote as $U_i^A(v_i, b_i, b_j; a_i)$ the utility of bidder i

$$U_i^A(v_i, b_i, b_j; a_i) = \begin{cases} v_i - b_i + a_i(b_i + b_j) & \text{if } b_i > b_j \\ -b_i + a_i(b_i + b_j) & \text{if } b_i < b_j \\ \frac{v_i}{2} - a_i)b_i + a_i(b_i + b_j) & \text{if } b_i = b_j \end{cases} \tag{1}$$

In contrast, in the first-price auction the bidder with the highest bid is the winner and

[5] Remark this is not true for the first-price auction.

pays her own bid while the loser does not pay anything. Contrary to the all-pay auction, here each bidder takes advantage of an externality only from the winner's bid which could be her own bid. Denote as $U_i^F(v_i, b_i, b_j; a_i)$; the utility of bidder i the follows

$$U_i^F(v_i, b_i, b_j; a_i) = \begin{cases} v_i - b_i + a_i b_i & \text{if } b_i > b_j \\ a_i b_j & \text{if } b_i < b_j \\ \frac{v_i}{2} - b_i + a_i b_i & \text{if } b_i = b_j \end{cases} \tag{2}$$

It is clear that the payment rule affects the returns that bidders obtain. In the all-pay auction, bidder i's utility is a function of her opponent's bid for each outcome of the auction. In the first-price auction, on the other hand, if the bidder i is the winner her payoff is independent of her opponent's bid.

Assumption (The limit of the bidders' altruism)**.** *Bidders are not fully altruistics.*

Indeed, they strictly prefer to keep their money for personal use rather than to spend it for the charitable purpose even if they win. The limit of the bidders' altruism is a consistent assumption.

In the all-pay auction, the limit of the bidders altruism leads to $\frac{\partial U_i^A}{\partial b_i}(v_i, b_i, b_j; a_i) < 0$ which is equivalent to $a_i < 1$. As bidders pay if they win as well as when they lose, the limit of the altruism leads to compute the derivatives of the bidders' utility in this two situations. Yet, the limit of the bidders' altruism is independent of the outcome of the auction. Thus, these derivatives runs to the same result.

In the first-price auction the limit of the bidders altruism leads to $\frac{\partial U_i^F}{\partial b_i}(v_i, b_i, b_j; a_i) < 0$ which is also equivalent to $a_i < 1$. As only the winner pays in the first-price auction only the outcome where he wins matter for the altruism level.

Bidder i's strategy is a function $\alpha(.; a_i) : [0,1] \to \mathbb{R}_+$ in the all-pay auction and a function $\beta(.; a_i) : [0,1] \to \mathbb{R}_+$ in the first-price auction which determines her bid for any value given her altruism parameter. Given a sensitivity level a_i different for each bidder, we focus on the asymmetric equilibria such that $\alpha(.; a_i) \equiv \alpha_i(.)$ and $\beta(.; a_i) \equiv \beta_i(.)$. However, as the bidders are distinguished only thanks to their altruism parameter, their equilibrium bidding functions would be symmetric in these parameters. Denote as $\varphi_i(.) = \alpha_i^{-1}(.)$ and $\phi_i(.) = \beta_i^{-1}(.)$ the inverse functions of bidder i's strategy functions given her altruism a_i. Notice that (α_i, α_j) is a Bayesian Nash equilibrium such that its meets the first and the second order conditions if and only if (φ_i, φ_j) also fulfill the first and the second order conditions. The same relationship also holds in the first-price auction with (β_i, β_j) and (ϕ_i, ϕ_j).

3 All-Pay Auction

As we said in the preliminary section, in the all-pay auction all bidders pay their own bid. Moreover, each bidder benefits an externality from her own bid as well as her from her competitor. Then, using (1) we can compute the expected payoff of bidder i

$$\mathbb{E}U_i^A(v_i, b_i, \alpha_j; a_i) = v_i\alpha_j^{-1}(b_i) - b_i + a_i(b_i + \mathbb{E}\alpha_j(V)). \tag{3}$$

To determine the effect of the altruism on the expected payoff we can divide (3) in two terms, the usual expected utility and the return from the charity purpose, κ_i^A. Then,

$$\mathbb{E}U_i^A(v_i, b_i, \alpha_j; a_i) = v_i\alpha_j^{-1}(b_i) - b_i + \kappa_i^A(b_i, \alpha_j; a_i)$$

with $\kappa_i^A(b_i, \alpha_j; a_i) = a_i(b_i + \mathbb{E}\alpha_j(V))$. Thus, if bidder i does not take account the term κ_i^A she would face the usual all-pay auction expected payoff.

Lemma 1. *The bidders' equilibrium strategies must be pure stragies that are continuous and increasing functions.*

Lemma 2. *Minimum and maximum bids must be the same for both bidders so that $\alpha_1(0) = \alpha_2(0) = 0$ and $\alpha_1(1) = \alpha_2(1) = \bar{b}$.*

In an all-pay auction, bidders care about their bids if they win as well as when they lose. In both cases, they get a positive return from their opponent's bid. Thus, their equilibrium bid depends of their own altruism parameter as well as on their competitor's. An immediate consequence of the Lemma 1 is that the inverse function of α_i, φ_i, is increasing and differentiable almost everywhere on $[0, \bar{b}]$. Furthermore, $\varphi_i(0) = 0$ and $\varphi_i(\bar{b}) = 1$ where $\bar{b} = \alpha_1(1) = \alpha_2(1)$.

To derive the equilibrium, we state here only the necessary condition while the sufficient condition is given in Appendix. Differentiating (3) with respect to b_i it follows that

$$\varphi_1(b) = \frac{1 - a_1}{\varphi_2'(b)} \text{ for all } b \in (0, \bar{b}] \tag{4}$$

$$\varphi_2(b) = \frac{1 - a_2}{\varphi_1'(b)} \text{ for all } b \in (0, \bar{b}]. \tag{5}$$

Then, from (4) and (5) and using the boundary conditions $\varphi_i(0) = 0$ we get

$$\varphi_i(b)\varphi_j(b) = (1 - a_i)b + (1 - a_j)b \text{ for all } b \in (0, \bar{b}]. \tag{6}$$

As $\varphi_i(\bar{b}) = 1$ for all i, $\bar{b} = \dfrac{1}{2 - a_1 - a_2}$ follows from (6). Then, for some level of the altruism parameters, bidders could submit a maximum bid higher than their valuation. Indeed, this would be the case if the sum of the altruism parameters is higher than 1. Moreover, if each altruism parameter is close to 1 the maximum bid would be infinite as in the case of symmetric bidders (see Goeree et al. (2005)). Thus, revenue is not bounded and could potentially be infinite.

Using (5), for $i = 1, 2$ equation (6) leads to

$$\varphi_i(b) = \frac{2 - a_j - a_j}{1 - a_j}\varphi_i'(b)b \text{ for all } b \in (0, \bar{b}].$$

From this we obtain an explicit solution of the inverse bid functions which characterize the unique Bayesian Nash equilibrium $(\varphi_1(.), \varphi_2(.))$:

$$\varphi_i(b) = [(2 - a_i - a_j)b]^{\frac{1-a_j}{2-a_i-a_j}} \text{ for all } b \in (0, \bar{b}], \text{ for } i = 1, 2 \tag{7}$$

Proposition 1. *There exists a unique Bayesian Nash equilibrium* (α_1, α_2) *such that*

$$\alpha_i(v) = \frac{1}{2 - a_i - a_j} v^{\frac{2-a_i-a_j}{1-a_j}} \quad \textit{for } i, j = 1, 2 \textit{ and } i \neq j$$

.

Obviously, for $a_1 = a_2 \equiv a$ we get the symmetric Nash equilibrium

$$\alpha_1(v) = \alpha_2(v) = \frac{1}{2(1-a)}v^2.$$

The equilibrium strategy function of bidder i is increasing in her own altruism parameter. Indeed, the more she is concerned with the charity purpose the higher her bid will be. On the other hand, the higher her opponent's sensitivity, the less she would like to bid. A higher sensitivity leads to a higher aggressiveness which affects her bid. These results can be verifies by computing the derivatives

$$\frac{\partial \alpha_i}{\partial a_i}(v; a_i, a_j) = -\frac{1}{(2 - a_i - a_j)^2} v^{\frac{2-a_i-a_j}{1-a_j}} \left(1 + \frac{2 - a_i - a_j}{1 - a_j} \ln v\right) \geq 0$$

$$\frac{\partial \alpha_i}{\partial a_j}(v; a_i, a_j) = \frac{-1 + (2 - a_i - a_j) v^{\frac{2-a_i-a_j}{1-a_j}} \frac{1-a_i}{(1-a_j)^2} \ln v}{(2 - a_i - a_j)^2} \leq 0$$

Figure 1 depicts the equilibrium bidding strategies for $a_1 = 0,75$ and $a_2 = 0,25$.

Corollary 1. *In the all-pay auction, the more altruistic bidder is the more agressive one. More precisely, if* $a_1 > a_2$ *then* $\alpha_1(v) > \alpha_2(v)$ *for all* $v \in (0, 1)$.

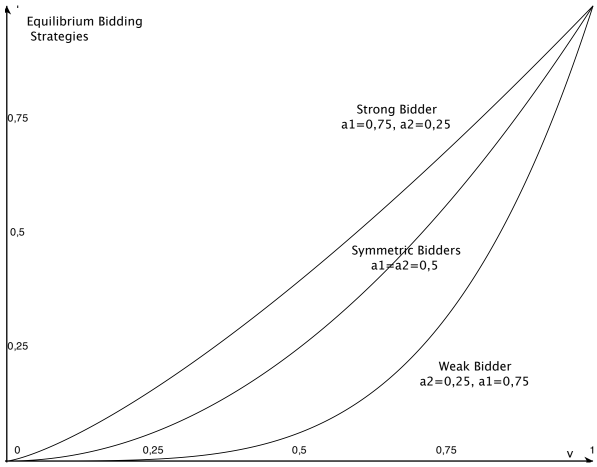

Figure 1: Equilibrium Bidding Strategies

4 First-Price Auction

In the first-price auction the bidder the highest bid gets the object and pays her own bid while the loser does not pay anything (see Section 2). Moreover, each bidder experiences a positive externality from the winner's bid. Using (2) we can then compute the expected payoff of bidder i

$$\mathbb{E}U_i^F(v_i, b_i, \beta_j; a_i) = [v_i - (1-a_i)b_i]\beta_j^{-1}(b_i) + a_i \int_{\beta_j^{-1}(b_i)}^{1} \beta_j(v)dv \tag{8}$$

$$= [v_i - (1-a_i)b_i]\beta_j^{-1}(b_i) + a_i \int_{b_i}^{\bar{b}_j} v(\beta_j^{-1})'(v)dv \tag{9}$$

$$= [v_i - b_i]\beta_j^{-1}(b_i) + a_i \left(\bar{b}_j - \int_{b_i}^{\bar{b}_j} \beta_j^{-1}(v)dv \right) \tag{10}$$

where $\bar{b}_j$ is bidder j's maximum bid. Define $y = \beta_j(v)$. With this, (9) follows from (8) and (10) is obtained through integration by parts. Again, we can split the expected payoff is two terms. The first one is the expected payoff of the usual first-price auction and the second the return from the charity purpose, κ_i^F:

$$[v_i - b_i]\beta_j^{-1}(b_i) + \kappa_i^F(b_i, \beta_j; a_i)$$

with $\kappa_i^F(b_i, \beta_j; a_i) = a_i \left(\bar{b}_j - \int_{b_i}^{\bar{b}_j} \beta_j^{-1}(v)dv \right)$. As before, if bidder i does not take account the term κ_i^F she would face the usual all-pay auction expected payoff.

Lemma 3. *The bidders' equilibrium strategies must be pure stragies that are continuous and increasing functions.*

Lemma 4. *Minimum bids are the same for both bidders while maximum bids are different so that* $\beta_1(0) = \beta_2(0) = 0$*,* $\beta_1(1) = \beta_2(1) = \bar{b}$ *and* $\bar{b} \in \left[\frac{1}{2}, 1\right)$*.*

Lemma 5. *Each bidder submit a non-negative bid inferior to her value such that* $\beta_i(v) < v$ *for all* $v \in (0, 1]$ *and* $i = 1, 2$*.*

As in the case of the all-pay auction, from the Lemma 3 the inverse function of β_i, ϕ_i, is increasing and differentiable almost everywhere on $[0, \bar{b}]$. Furthermore, $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(\bar{b}) = \phi_2(\bar{b}) = 1$. Bidders could not submit a maximum bid higher than their valuation. Furthermore, the maximum bid is bounded because of the limit on the bidders' altruism. The maximum bid in the all-pay auction is therefore higher than the one in the first-price auction.[6]

To derive the equilibrium, as above we give only state the necessary condition while the sufficient condition is given in Appendix. Differentiating (8) with respect to b_i it follows

$$\phi_1'(b) = \frac{1 - a_2}{\phi_2(b) - b}\phi_1(b) \text{ for all } b \in (0, \bar{b}] \tag{11}$$

$$\phi_2'(b) = \frac{1 - a_1}{\phi_1(b) - b}\phi_2(b) \text{ for all } b \in (0, \bar{b}]. \tag{12}$$

There is no explicit solution to this differential equation systems with our bounded conditions. The equations (11) and (12) and the boundary conditions define equilibrium strategies if they define the optimal decision for each bidder.

Proposition 2. *The unique Bayesian Nash equilibrium* (β_1, β_2) *is characterized by the inverse bidding functions* (ϕ_1, ϕ_2) *such that*

$$\phi_i'(b) = \frac{1 - a_j}{\phi_j(b) - b}\phi_i(b) \textit{ for all } b \in (0, \bar{b}]$$

which satisfies the boundary conditions $\beta_i(0) = 0, \beta_i(1) = \bar{b}$*, for* $i = 1, 2$ *and* $i \neq j$*.*

For $a_1 = a_2 \equiv a$ we get the symmetric Nash equilibrium (see Engers and McManus (2007) for details) such that $\beta_i(v) = \dfrac{v^2}{2 - a}$ for $i = 1, 2$. The maximum bid is bounded and then suggests that bids are bounded as well as the expected revenue. As in the all-pay auction we can established a strict ranking of the bidding functions.

Corollary 2. *In the first-price auction, the more altruistic bidder is the more agressive one. More precisely, if* $a_1 > a_2$ *then* $\beta_1(v) > \beta_2(v)$ *for all* $v \in (0, 1)$*.*

This result is useful to determine the shape of the bidding strategies at the equilibrium. Indeed, β_1 and β_2 cannot intersect. Then, the equilibrium bidding strategies are concave for bidder 1 and convex for bidder 2.[7] Figure 2 depicts the curves of β_1 and β_2.

[6] This result is not obvious as for some value of the altruism parameters the maximum bid in the all-pay auction is inferior to 1. Claim 1 establishes this result in the Appendix.

[7] A proof is given in the Appendix, Claim 3.

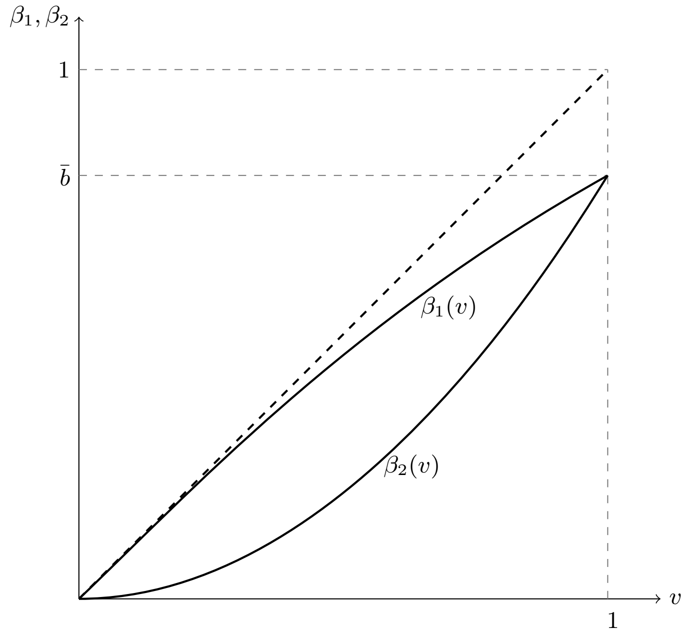

Figure 2: Equilibrium Bidding Strategies

5 Revenue Comparisons

In this section we examine the performance of the all-pay and first-price auctions in terms of the expected revenue. As before we assume that bidder 1 is more concerned by the charity purpose that bidder 2 which means that $a_1 > a_2$. Our next result describes the ranking of the equilibrium bidding strategies for each bidder.

Proposition 3. *The bidders' i bidding strategies in the all-pay and the first price auction intersect only once such that*

$$\beta_i(v) \geq \alpha_i(v) \text{ for all } v \in [0, \bar{v}_i] \text{ and } \alpha_i(v) > \beta_i(v) \text{ for all } v \in (\bar{v}_i, 1], \text{ for } i = 1, 2 \text{ and } i \neq j.$$

Let us denote e_i^A and e_i^F the expected payment of the bidder i in the all-pay and first-price auctions. These expected payments are $e_i^A(v) = \alpha_i(v)$ and $e_i^F(v) = v\beta_i(v)$ for all $v \in [0, 1]$. Using the Proposition 3, we can compare the expected payments and potentially rank them.

Lemma 6. *The expected payment of the bidder 1 from the all-pay auction is greater than her expected payment from the first-price auction.*

Lemma 7. *The expected payment of the bidder 2 from the all-pay auction is greater than her expected payment from the first-price auction if her valuation is sufficiently high. Moreover, her expected payment is the same in both auctions if her valuation is sufficiently low.*

Then, if it is not clear if the expected payment of bidder 2 from the all-pay auction is greater than from the first-price auction. Indeed, as both are convex functions it could happen that for a range of middle valuations the latter outperforms the former. The next proposition determines the ranking of the expected revenue.

Proposition 4. *If the bidders' altruism parameters for the charity purpose are non-negative, the expected revenue in the all-pay auction is strictly higher than in the first-price auction.*

Thus, the introduction of the asymmetry on the altruism parameters does not change the ranking of the expected revenue (Goeree et al., 2005, Engers and McManus, 2007). This result was not predictable as the asymmetry can reverse the ranking of the expected revenue in first and second-price auctions (Bulow et al., 1999). Furthermore, that contradicts results with complete information (Bos, 2008). Then, our result confirms the dominance of the all-pay auction at raising money for charity in an incomplete information framework.

Moreover, the expected revenue in the all-pay auction is given by

$$\begin{aligned}\mathbb{E}R^A(a_1, a_2) &= \int_0^1 \alpha_1(v)dv + \int_0^1 \alpha_2(v)dv \\ &= \frac{1}{2 - a_1 - a_2}\left(\frac{1 - a_2}{3 - a_1 - 2a_2} + \frac{1 - a_1}{3 - 2a_1 - a_2}\right)\end{aligned}$$

It is interesting to see how asymmetry affects the expected revenues in the all-pay auction. Let us denote $\bar{a} = a_1 + a_2$, such as $\bar{a} \in [0, 2)$. Upon substitution, we can see that $\mathbb{E}R^A(a_1, \bar{a} - a_1)$ is maximized at $a_1 = \frac{\bar{a}}{2}$ and then increasing for $a_1 < \frac{\bar{a}}{2}$ and decreasing for $a_1 > \frac{\bar{a}}{2}$. For example, Figure 3 depicted the situation when $\bar{a} = 1$. Then, we get the following results,

Lemma 8. *The greater asymmetry in the altruism parameters for the charity purpose the higher the expected revenue will be in the all-pay auction.*

This result is online with results on asymmetric all-pay auctions with complete information. Hillman and Riley (1989) determine that the expected revenue decreases when the bidders become more asymmetric.

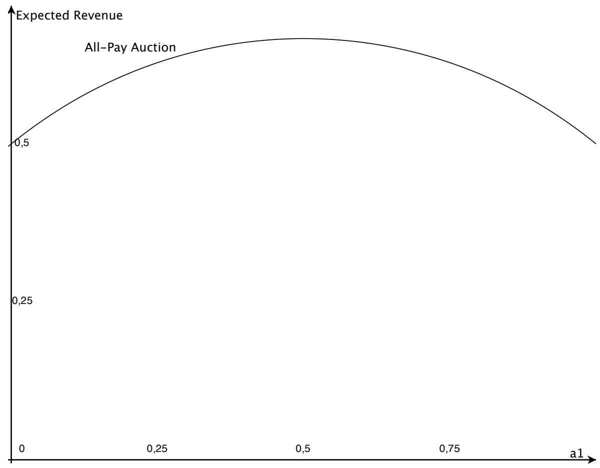

Figure 3: Expected Revenue of the All-Pay Auction for $\bar{a} = 1$.

Unfortunately, as we do not have explicit bidding functions in the first-price auction we cannot provide the expected revenue for this design and determine how the asymmetry affects it.

6 Conclusion

The purpose of this paper was to determine which of the two auction designs – all-pay auction or first-price auction – is better at raising money for charity when bidders are asymmetric on their altruism parameters with complete information and values are drawn in a independent private values model. As in the case with symmetric bidders (Goeree et al., 2005) we conclude that the all-pay auction is better than the first-price auction. These results show that different auction designs are better for different environments. Indeed, in a complete information framework Bos (2008) shows first-price auctions outperform all-pay auctions when the asymmetry among bidders is strong enough. Moreover, Carpenter et al. (2007) conclude there is no strict ranking of revenue when the participation is endogenous.

Our result confims the one of Goeree et al. (2005) and indicates that all-pay auctions should be considered seriously to raise money for charity purposes. As we pointed out, the organization of an all-pay is unproblematic. A one-shot sale of tickets with the winner being determined by the highest number of tickets bought is equivalent to an all-pay auction.

This paper and more generally the idea that the optimal auction design for charity depends on the informational setup is good candidate for experiments in a lab. In this way one could expect to determine which elements in the knowledge of bidders are crucial to the ranking of auctions by revenue.

7 Appendix

The derivation of statements in lemmata 1 and 3 uses similar arguments than in de Frutos (2000).

Proof of the Lemma 1. First, let us show that the equilibrium bidding strategies are monotonically increasing. Denote, for a fixed a_i, $\bar{b} = \alpha_i(\bar{v})$ and $\underline{b} = \alpha_i(\underline{v})$ with $\bar{v} \geq \underline{v}$. Then, at the equilibrium, we should get

$$\begin{aligned} \mathbb{E}U_i^A(\bar{v}, \bar{b}, \alpha_j; a_i) &\geq \mathbb{E}U_i^A(\bar{v}, \underline{b}, \alpha_j; a_i) \\ \mathbb{E}U_i^A(\underline{v}, \underline{b}, \alpha_j; a_i) &\geq \mathbb{E}U_i^A(\underline{v}, \bar{b}, \alpha_j; a_i) \end{aligned}$$

which could be written

$$\begin{aligned} \bar{v}\alpha_j^{-1}(\bar{b}) - (1-a_i)\bar{b} + a_i\mathbb{E}\alpha_j(V) &\geq \bar{v}\alpha_j^{-1}(\underline{b}) - (1-a_i)\underline{b} + a_i\mathbb{E}\alpha_j(V) \\ \underline{v}\alpha_j^{-1}(\underline{b}) - (1-a_i)\underline{b} + a_i\mathbb{E}\alpha_j(V) &\geq \underline{v}\alpha_j^{-1}(\bar{b}) - (1-a_i)\bar{b} + a_i\mathbb{E}\alpha_j(V). \end{aligned}$$

Then, substracted the second inequality the the first one runs to $(\bar{v} - \underline{v})(\alpha_j^{-1}(\bar{b}) - \alpha_j^{-1}(\underline{b})) \geq 0$. Then, $\underline{b} \leq \bar{b}$.

Let us assume there is a gap $[b', b'']$ in $\alpha_i(.)$. Then, if bidder j planned to submit a bid in (b', b'') he would strictly prefer to bid b'. Indeed, it does not affect her probability of winning and decreases her payment. Consequently, bidding b'' for bidder i is dominated by bidding $b' + \varepsilon$ with $\varepsilon > 0$. Thus the equilibrium bidding strategies are without any gap.

Let us consider there is a atom in $\alpha_i(.)$ such as it exists b' with $\mathbb{P}(\alpha_i(v_i) = b') > 0$. Then there is an $\varepsilon > 0$ such that there is a gap $(b' - \varepsilon, b')$ in $\alpha_i(.)$. Thus, it follows a contradiction with the last paragraph.

As the equilibrium bidding strategies are without any atom and monotonically increasing, they are strictly monotonically increasing. Furthermore the equilibrium bidding strategies are in pure strategies as there is no gap. Then, the equilibrium strategies are differentiable almost everywhere. ■

Proof of the Lemma 2. Assume that $0 \leq \alpha_i(0) \leq \alpha_j(0)$. Each bidder gets the same payoff by winning as well as losing. As bidders have a strict preference for a higher payoff independently of the outcome, it follows that $\alpha_i(0) = \alpha_j(0) = 0$. Assume that $\alpha_j(1) > \alpha_i(1)$. Then, the bidder 1 can decrease her bid without alterate her winning probability and increasing her payoffs. Similarly, $\alpha_i(1) > \alpha_j(1)$ cannot be part of the equilibrium. Thus, $\alpha_1(1) = \alpha_2(1)$. ■

Proof of the Proposition 1. It is clear that at the equilibrium $\alpha_i(0) = 0$. Indeed, if $b_i = 0$ the payoff of the bidder i for $v_i > 0$ is strictly inferior to the one for $v_i = 0$. Consider now the payoff of the bidder i for all $b_i \in (0, \bar{b}]$.

$$\begin{aligned} \frac{\partial U_i^A}{\partial b_i}(v_i, b_i, \alpha_j; a_i) &= v_i\varphi_j'(b_i) - (1-a_i) \\ &= (v_i - \varphi_i(b_i))\varphi_j'(b_i). \end{aligned}$$

To get the last line we used the necessary condition $\varphi_i(b_i)\varphi_j'(b_i) = 1 - a_i$. When $v_i > \varphi_i(b_i)$ it follows that $\frac{\partial U_i^A}{\partial b_i}(v_i, b_i, \alpha_j; a_i) > 0$. In a similar manner, when $v_i < \varphi_i(b_i)$, $\frac{\partial U_i^A}{\partial b_i}(v_i, b_i, \alpha_j; a_i) < 0$. Thus, $\frac{\partial U_i^A}{\partial b_i}(v_i, \alpha_i, \alpha_j; a_i) = 0$. As a result, the maximum of $U_i^A(v_i, \alpha_i, \alpha_j; a_i)$ is achieved for $v_i = \varphi_i(b_i)$ and then $b_i = \alpha_i(v_i)$. ∎

Proof of the Corollary 1. Remind that we assume $a_1 > a_2$. As $\alpha_i(x) \in [0, 1]$ for all i and all x. Then we get $\varphi_1(x) < \varphi_2(x)$ for all x. The result follows. ∎

Proof of the Lemma 3. First, let us show that the equilibrium bidding strategies are monotonically increasing. Denote, for a fixed a_i, $\bar{b} = \beta_i(\bar{v})$ and $\underline{b} = \beta_i(\underline{v})$ with $\bar{v} \geq \underline{v}$. Then, as for the all-pay auction at the equilibrium, we should get

$$\begin{aligned}
&(\bar{v} - (1 - a_i)\bar{b})\beta_j^{-1}(\bar{b}) + a_i \int_{\beta_j^{-1}(\bar{b})}^{1} \beta_j(v)dv \geq (\bar{v} - (1 - a_i)\underline{b})\beta_j^{-1}(\underline{b}) + a_i \int_{\beta_j^{-1}(\underline{b})}^{1} \beta_j(v)dv \\
&(\underline{v} - (1 - a_i)\underline{b})\beta_j^{-1}(\underline{b}) + a_i \int_{\beta_j^{-1}(\underline{b})}^{1} \beta_j(v)dv \geq (\underline{v} - (1 - a_i)\bar{b})\beta_j^{-1}(\bar{b}) + a_i \int_{\beta_j^{-1}(\bar{b})}^{1} \beta_j(v)dv
\end{aligned}$$

Then we obtain $(\bar{v} - \underline{v})(\beta_j^{-1}(\bar{b}) - \beta_j^{-1}(\underline{b})) \geq 0$. Then, $\underline{b} \leq \bar{b}$. With similar arguments that in Lemma 1 the equilibrium bidding strategies must be gapless and atomless. In consequences, the equilibrium bidding strategies are in pure strategies and strictly monotonically increasing. Then, the equilibrium strategies are differentiable almost everywhere. ∎

Proof of the Lemma 4 and 5. Assume that $\beta_i(0) < \beta_j(0)$. When the valuation is 0, the payoff of losing is higher than the payoff of winning. Then, both bidders deviate and submit a bid equal to 0 such that $\beta_1(0) = \beta_2(0) = 0$.

Assume that $\bar{b}_i > \bar{b}_j$. Then bidder i wins for sure and get an expected payoff $1 - b_i$. As $\bar{b}_i > \bar{b}_j$, she could increase her expected payoff without alterate her probability of winning by decreasing her bid to $\bar{b}_j$. It follows that $\bar{b}_i = \bar{b}_j = \bar{b}$. Furthermore, we determine that a bidder will never submit an equilibrium bid higher than her valuation v. To see it, compare issues where bidder i with a valuation v, either bids $b = v$ or $b = v + \varepsilon$ with $\varepsilon > 0$. Using (8) it follows,

$$\begin{aligned}
U_i^F(v, v, \beta_j; a_i) - U_i^F(v, v + \varepsilon, \beta_j; a_i) &= a_i v(\beta_j^{-1}(v) - \beta_j^{-1}(v + \varepsilon)) + (1 - a_i)\varepsilon\beta_j^{-1}(v + \varepsilon) \\
&\quad + a_i \int_{\beta_j^{-1}(v)}^{\beta_j^{-1}(v+\varepsilon)} \beta_j(x)dx \\
&= (1 - a_i)\varepsilon\beta_j^{-1}(v + \varepsilon) + a_i \int_{\beta_j^{-1}(v)}^{\beta_j^{-1}(v+\varepsilon)} \beta_j(x) - v dx
\end{aligned}$$

For all $x \in [\beta_j^{-1}(v), \beta_j^{-1}(v+\varepsilon)]$ $\beta_j(x) - v \geq 0$. Hence, $U_i^F(v, v, \beta_j; a_i) - U_i^F(v, v+\varepsilon, \beta_j; a_i) > 0$. Thus, $\beta_i(v) \leq v$ for all $v \in [0, 1]$ and $\bar{b} \leq 1$. It follows that $\phi_i(b) \geq b$. In addition, $\phi_i(0) = 0$

and $\phi_i(.)$ is strictly increasing then $\phi_i(b) > 0$ for all $b > 0$. In consequences, we get $\phi_i(b) > b$ from (11) and (12). Hence, $\bar{b} < 1$.

Summing these differential equations (11) and (12) it follows

$$\phi_1'(b)\phi_2(b) + \phi_2'(b)\phi_1(b) - b(\phi_1'(b) + \phi_2'(b)) - (\phi_1(b) + \phi_2(b)) = -a_1\phi_2(b) - a_2\phi_1(b)$$

Intregrating this equation and using $\phi_i(\bar{b}) = 1$,

$$1 - 2\bar{b} = -\int_0^{\bar{b}} a_1\phi_2(x) + a_2\phi_1(x)dx$$

Hence, $\bar{b} \geq \frac{1}{2}$. ∎

Proof of the Proposition 2. It is clear that at the equilibrium $\beta_i(0) = 0$. Indeed, if $b_i = 0$ the payoff of the bidder i for $v_i > 0$ is strictly inferior to the one for $v_i = 0$. Consider now the payoff of the bidder i for all $b_i \in (0, \bar{b}_i]$.

$$\begin{aligned}\frac{\partial U_i^F}{\partial b_i}(v_i, b_i, \beta_j; a_i) &= (v_i - b_i)\phi_j'(b_i) - (1 - a_i)\phi_j(b_i) \\ &= \frac{v_i - b_i}{\phi_i(b_i) - b_i}(1 - a_i)\phi_j(b_i) - (1 - a_i)\phi_j(b_i)\end{aligned}$$

To get the last line we used the necessary condition provide by equations (11) and (12). $\phi_i(b_i)\phi_j'(b_i) = 1 - a_i$. When $v_i > \phi_i(b_i)$ it follows that $\frac{\partial U_i^F}{\partial b_i}(v_i, b_i, \beta_j; a_i) > 0$. In a similar manner, when $v_i < \phi_i(b_i)$, $\frac{\partial U_i^F}{\partial b_i}(v_i, b_i, \beta_j; a_i) < 0$. Thus, $\frac{\partial U_i^F}{\partial b_i}(v_i, \beta_i, \beta_j; a_i) = 0$. As a result, the maximum of $U_i^F(v_i, \beta_i, \beta_j; a_i)$ is achieved for $v_i = \phi_i(b_i)$ and then $b_i = \beta_i(v_i)$. ∎

Proof of the Corollary 2. Remark that if $\exists\ y \in (0, \bar{b})$ and $\phi_1(y) = \phi_2(y) = z$, then (10) and $a_1 > a_2$ imply that

$$\phi_2'(y) = \frac{1 - a_1}{z - y}z < \phi_1'(y) = \frac{1 - a_2}{z - y}z$$

Hence, thanks to properties of the inverse functions, if there exists a z such that $\beta_1(z) = \beta_2(z) = y$ then $\beta_2'(z) > \beta_1'(z)$. In consequence, β_1 and β_2 intersect at most once.

To prove the result let us assume the contrary. Suppose $\exists x \in (0, 1)$ such that $\beta_2(x) \geq \beta_1(x)$. Then either $\beta_2(v) > \beta_1(v)$ for all $v \in (0, 1)$ or they intersect in $z \in (0, 1)$ and for all $x \in (z, 1)$, $\beta_2(x) > \beta_1(x)$. In the latter case, $\phi_2(b) < \phi_1(b)$ for all b close to $\bar{b}$. Notice that from (11) and (12) it follows

$$\phi_1(b) = \frac{\phi_2(b)}{\phi_2'(b)}(1 - a_1) + b \text{ and } \phi_2(b) = \frac{\phi_1(b)}{\phi_1'(b)}(1 - a_2) + b.$$

Using $a_1 > a_2$ and $\phi_1(b) > \phi_2(b)$ we obtain $\dfrac{\phi_2(b)}{\phi_2'(b)} > \dfrac{\phi_1(b)}{\phi_1'(b)}$ for b close to $\bar{b}$. Therefore, $\phi_2(b) > \phi_1(b)$; hence a contradiction.[8] ■

Proof of the Proposition 3.

Claim 1. The maximum bid in all-pay auction is higher than is first-price auction for non-negative altruism parameters.

Proof. Let us denote $\bar{b}^A$ and $\bar{b}^F$ the maximum bids in the all-pay and first-price auction. Clearly, $\bar{b}^A \geq 1 > \bar{b}^F$ for all $a_1 + a_2 \geq 1$. Let us assume that $\bar{b}^F \geq \bar{b}^A$ for some $a_1 + a_2 < 1$. Then, by continuity there exists a value of $a_1 + a_2$ such that $\bar{b}^F = \bar{b}^A$. If this case happens with asymmetric bidders that also happens with symmetric bidders. In the latter case, $a_1 + a_2 = a$, $\bar{b}^F = \dfrac{1}{2-a}$ and $\bar{b}^A = \dfrac{1}{2(1-a)}$. Hence the result.

As $\bar{b}^A > \bar{b}^F$ and the bidding strategies are strictly increasing functions, there exists $\bar{v}_i \in (0,1)$ such that $\alpha_i(\bar{v}_i) = \bar{b}^F$ for $i = 1, 2$. Then, $\alpha_i(v) > \beta_i(v)$ for all $v \in [\bar{v}_i, 1]$ for $i = 1, 2$. Hence, $\varphi_i(\bar{b}^F) < \phi_i(\bar{b}^F)$ for $i = 1, 2$.

Claim 2. $\varphi_i(b) > \phi_i(b)$ and $\varphi_j(b) > \phi_j(b)$ for all b close to 0.

Proof. Using L'Hôspital's rule in (11) implies:

$$\begin{aligned} 1 - a_i &= \lim_{b \to 0} \phi_j'(b) \frac{\phi_i(b) - b}{\phi_j(b)} \\ &= \phi_j'(0) \lim_{b \to 0} \frac{\phi_i(b) - b}{\phi_j(b)} \\ &= \phi_j'(0) \lim_{b \to 0} \frac{\phi_i'(b) - 1}{\phi_j'(b)} \\ &= \phi_i'(0) - 1 \end{aligned}$$

Thus, $\phi_i'(0) = 2 - a_i$ for $i = 1, 2$.

As $\varphi_i'(b) = (1 - a_j)((2 - a_i - a_j)b)^{\frac{-1+a_i}{2-a_i-a_j}}$, and $a_i > a_j$, $\lim_{b \to 0} \varphi_i'(b) = +\infty$. Hence, $\varphi_i'(0) > \phi_i'(0)$ for $i = 1, 2$. Therefore, $\varphi_i(b) > \phi_i(b)$ for all b sufficiently close to 0 and $\beta_i(v) > \alpha_i(v)$ for all v sufficiently close to 0.

Claim 3. The inverse bidding strategies ϕ_1 and ϕ_2 are respectively convex and concave functions.

Proof. Remark that from (11) and (12) ϕ_1 and ϕ_2 are continuous functions and then differentiable. From (11) and (12) we obtain

$$\phi_i(b)'' = \frac{1 - a_j}{(\phi_j(b) - b)^2} (\phi_i'(b)(\phi_j(b) - b) - (\phi_i(b)(\phi_j'(b) - 1)) \text{ for } i = 1, 2 \text{ and } i \neq j. \quad (13)$$

[8] As $\phi_i(\bar{b}) = 1, \phi_i(0) = 0$ and $\phi_i'(b) > 0$ it follows that $\dfrac{\phi_2(b)}{\phi_2'(b)} > \dfrac{\phi_1(b)}{\phi_1'(b)}$ implies $\phi_2(b) > \phi_1(b)$. That can be shown as the dominance in terms of the reverse hazard rate implies the stochastic dominance (see Krishna (2002) for an example of the proof).

Let us assume that $\phi_2''(b) > 0$ for all $b \in [0, \bar{b}^F]$. Remark that $\phi_1''(b) < 0$ is equivalent to $\frac{\phi_1'(b)}{\phi_1(b)} < \frac{\phi_2'(b) - 1}{\phi_2(b) - b}$ Using (11), this is also equivalent to $\phi_2'(b) > 2 - a_2$. Thus, as $\phi_2'(0) = 2 - a_2$ ϕ_2 convex leads to ϕ_1 concave. Yet, ϕ_1 concave, ϕ_2 convex and the boundary conditions contradict the Corollary 2. Hence, ϕ_2 cannot be convex.

Let us assume that ϕ_2 is neither convex nor concave. Then there exists at least one inflexion point b such as $\phi_2''(b) = 0$. Denote $\tilde{b}$ the first inflexion point. Then, $\phi_2''(\tilde{b}) = 0$ and (13) imply $\phi_1'(\tilde{b}) = 2 - a_1$. As $\phi_1'(0) = 2 - a_1$, ϕ_1' is not strictly monotone on $[0, \tilde{b}]$ and there exists $\tilde{\tilde{b}}$ such as $\phi_1''(\tilde{\tilde{b}}) = 0$ with $\tilde{\tilde{b}} < \tilde{b}$.[9] In the same way, $\phi_1''(\tilde{\tilde{b}}) = 0$ and (13) imply $\phi_2'(\tilde{\tilde{b}}) = 2 - a_2$. As $\phi_2'(0) = 2 - a_2$, ϕ_2' is not monotone on $[0, \tilde{\tilde{b}}]$ which contradicts that $\tilde{b}$ is the first inflexion point of ϕ_2.[10] Hence, ϕ_2 has to be either convex or concave. With a symmetric argument we get the same result for ϕ_1.

In consequence $\phi_2''(b) \leq 0$ for all $b \in [0, \bar{b}^F]$. Furthermore, $\phi_1''(b) \geq 0$ if and only if $2 - a_2 \geq \phi_2'(b)$ which is true as ϕ_2 concave and $\phi_2'(0) = 2 - a_2$. Hence, ϕ_1 is convex.

Claim 4. The inverse bidding strategy φ_i is a concave function.

Proof. Differentiating twice (7) leads to $\varphi_i''(b) = -(1 - a_j)(1 - a_i)((2 - a_i - a_j)b)^{\frac{-3+2a_i+a_j}{2-a_i-a_j}}$ for all $b \in [0, \bar{b}^A]$, which is negative.

Claim 2–4 imply that the curves ϕ_i and φ_i intersect once and only once. Moreover, $\varphi_i(b) \geq \phi_i(b)$ for all $b \in [0, \tilde{b}_i]$ with $\tilde{b}_i < \bar{b}^F$ and $\varphi_i(b) < \phi_i(b)$ for all $b \in [\tilde{b}_i, \bar{b}^F]$. Furthermore, we have shown that $\alpha_i(v) > \beta_i(v)$ for all $v \in [\bar{v}_i, 1]$ with $\alpha_i(\bar{v}_i) = \bar{b}^F$. Hence, the result. ■

Proof of the Lemma 6. First, remark that $e^A(0) = e^F(0) = 0$. Moreover, prove $e_1^A(v) > e_1^F(v)$ for $b \in (0, 1]$ is equivalent to prove $\frac{1}{v}\alpha_1(v) > \beta_1(v)$ for all $v \in (0, 1]$. Let us define $\tilde{\alpha}_1(v) = \frac{1}{v}\alpha_1(v)$ and $\tilde{\varphi}_1(b)$ the inverse function of $\tilde{\alpha}_1$ such that $\tilde{\varphi}_1(b) = ((2 - a_1 - a_2)b)^{\frac{1-a_2}{1-a_1}}$. Then, $\tilde{\varphi}_1$ is an increasing function and $\tilde{\varphi}_1(0) = 0$.

We have shown that there $\bar{v}_1 \in (0, 1)$ such that $\alpha_1(\bar{v}_1) = \bar{b}^F$ and $\alpha_1(v) > \beta_1(v)$ for all $v \in [\bar{v}_1, 1]$. Then there exists $\tilde{v}_1$ inferior to $\bar{v}_1$ such that $\tilde{\alpha}_1(\tilde{v}_1) = \bar{b}^F$ and $\tilde{\alpha}_1(v) > \beta_1(v)$ for all $v \in [\tilde{v}_1, 1]$.

Now, let us show that $\tilde{\alpha}_1(v) > \beta_1(v)$ for all $v \in (0, \tilde{v}_1]$. As $\alpha_1(\tilde{v}_1) = \beta_1(1) = \bar{b}^F$ we obtain $\tilde{\varphi}_1(\bar{b}^F) < \phi_1(\bar{b}^F)$. Moreover, as $\tilde{\varphi}_1'(b) = \frac{1-a_2}{1-a_1}(2 - a_1 - a_2)^{\frac{1-a_2}{1-a_1}} b^{\frac{a_1-a_2}{1-a_2}}$ and $a_1 > a_2$, we obtain $\tilde{\varphi}_1'(0) = 0$. Hence, $\phi_1'(0) > \tilde{\varphi}_1'(0)$. Therefore, $\phi_1(b) > \tilde{\varphi}_1(b)$ for all b sufficiently close to 0. Moroever, $\tilde{\varphi}_1''(b) = \frac{a_1-a_2}{1-a_2}\frac{1-a_2}{1-a_1}(2 - a_1 - a_2)^{\frac{1-a_2}{1-a_1}} b^{\frac{-1+a_1}{1-a_2}} \geq 0$. Thus, functions ϕ_1 and φ_1 are increasing, convex, $\phi_1(\bar{b}^F) > \tilde{\varphi}_1(\bar{b}^F)$ and $\phi_1'(0) > \tilde{\varphi}_1'(0)$ which leads to $\phi_1(b) > \tilde{\varphi}_1(b)$ for all $b \in (0, \bar{b}^F]$. Hence, the result. ■

[9] Remark that if ϕ_1' is constant on $[0, \tilde{b}]$, ϕ_2' is also constant on this interval and $\tilde{b}$ cannot be an inflexion point.

[10] Remark that if ϕ_2' is constant on $[0, \tilde{\tilde{b}}]$, ϕ_1' is also constant on this interval. Thus, $\tilde{\tilde{b}}$ cannot be an inflexion point for ϕ_1.

Proof of the Lemma 7. The expected payment of the bidder 2 from the first-price auction is given by $e_2^F(v) = v\beta_2(v)$. Then, $e_2^F(0) = 0$, $e_2^F(1) = \bar{b}^F$, $e_2^F{}'(v) = \beta_2(v) + v\beta_2'(v)$ and $e_2^F{}''(v) = 2\beta_2'(v) + v\beta_2''(v)$. As β_2 is increasing and convex, e_2^F is also increasing and convex. Moreover, $e_2^F{}'(0) = \alpha_2'(0)$ and $e_2^F(1) < \alpha_2(1)$. Hence, the result. ■

Proof of the Proposition 4. Before to show the result, let us establish the inequality (14).

Claim 5.

$$\int_0^1 \frac{x^2}{2}\beta_i'(x)dx \geq \int_0^1 \frac{x^2}{2}dx \int_0^1 \beta_i'(x)dx \tag{14}$$

Proof. β_2' is an increasing function. Then, for $i = 2$ (14) is a special case of the Chebyshev's inequality for monotone functions. Yet, this inequality cannot be applied for $i = 1$ as β_1' is decreasing. However, (14) is equivalent to $\displaystyle\int_0^1 \frac{x^2}{2}(\beta_1'(x) - \bar{b}^F)dx$. Then, let us show that $\beta_1'(x) \geq \bar{b}^F$ for all $x \in [0,1]$. Moreover, $\beta_1'(x) \geq \beta_1'(1)$ and $\bar{b}^F \leq \dfrac{1}{2-a_2}$ as β_1' is decreasing and the maximum bid with asymmetric bidders cannot be higher than the maximum bid with symmetric bidders. Therefore, we need to establish that $\beta_1'(1) \geq \dfrac{1}{2-a_2}$. Suppose the contrary which is equivalent to $\phi_1(\bar{b}^F)' \geq 2 - a_2$. This inequality is also equivalent to $\dfrac{1-a_1}{1-\bar{b}^F} \geq 2 - a_2$ which leads to $\bar{b}^F \geq \dfrac{1-a_2+a_1}{2-a_2}$. As $\dfrac{1-a_2+a_1}{2-a_2} > \dfrac{1}{2-a_2}$ we obtain $\bar{b}^F > \dfrac{1}{2-a_2}$; hence a contradiction.

Denote Δ_i the difference among $\displaystyle\int_0^1 e_i^A(v)dv$ and $\displaystyle\int_0^1 e_i^F(v)dv$. Then,

$$\begin{aligned}
\Delta_i &= \int_0^1 (\alpha_i(v) - v\beta_i(v))dv \\
&= \bar{b}^A - \int_0^1 v\alpha_i(v)dv - \frac{\bar{b}^F}{2} + \int_0^1 \frac{v^2}{2}\beta_i'(v)dv && (15)\\
&\geq \bar{b}^A - \int_0^1 v\alpha_i(v)dv - \frac{\bar{b}^F}{3} && (16)\\
&\geq \frac{1}{2-a_1-a_2} - \frac{1}{3-2a_j-a_i} - \frac{1}{3(2-a_2)} && (17)
\end{aligned}$$

Integrating by parts we obtain (15). Using the Claim 5, (15) leads to (16). To get (17) we use the fact that the maximum bid with asymmetric bidders cannot be higher than the maximum bid with symmetric bidders. Then it follows

$$\Delta_1 = \frac{5a_1 - a_1^2 - 3a_1a_2 - 2a_2 + a_2^2}{3(2-a_2)(2-a_1-a_2)(3-a_1-2a_2)}$$
$$\Delta_2 = \frac{a_1 - 2a_1^2 + 2a_2 - a_2^2}{3(2-a_2)(2-a_1-a_2)(3-2a_1-a_2)}$$

and $\Delta_1 + \Delta_2 \geq \dfrac{\delta(a_1,a_2)}{3(2-a_2)(2-a_1-a_2)(3-2a_1-a_2)(3-a_1-2a_2)}$ with $\delta(a_1,a_2) = (3-a_1-2a_2)(a_1-2a_1^2+2a_2-a_2^2)+(3-2a_1-a_2)(5a_1-a_1^2-3a_1a_2-2a_2+a_2^2)$. Let us show that the function $\delta(a_1,a_2)$ is positive for all a_1 given a_2 fixed and $a_1 > a_2$.

First, remark that for each value of a_2 inferior to a_1, the minimum and the maximum of the function δ are given by $\delta(a_2, a_2) = 18(-1 + a_2)^2 a_2 > 0$ and $\delta(1, a_2) = 2 - 3a_2 + a_2^3 > 0$. Moreover, $\frac{\partial \delta}{\partial a_1}(a_1, a_2) = 2[6a_1^2 + a_1(11a_2 - 20) + 9 - 7a_2 + a_2^2]$. Then, study the monoticity of δ given a_2 consists to study the sign of the polynomial

$$6a_1^2 + a_1(11a_2 - 20) + 9 - 7a_2 + a_2^2 \tag{18}$$

The discriminant of the equation (18) is $85a_2^2 - 188a_2 + 76$ and thus non-positive for all $a_2 > \underline{a}_2 \equiv \frac{94 - 2\sqrt{594}}{85} \sim 0,532$. Therefore, for all $a_1 \in (\underline{a}_2, 1)$ given $a_2 > \underline{a}_2$ the function δ is increasing in a_1. Hence, $\Delta_1 + \Delta_2 > 0$.

Yet, when $a_2 \leq \underline{a}_2$ the equation (18) could positive as well as negative. Indeed, (18) is positive for all $a_1 \leq \underline{a}_1$ and non-positive for all $a_1 > \underline{a}_1$ with $\underline{a}_1 \equiv \frac{20 - 11a_2 + \sqrt{85a_2^2 - 188a_2 + 76}}{12}$. Remark that $\underline{a}_1$ is positive but superior to 1 when $a_2 > \tilde{a}_2 \equiv \frac{-1 + \sqrt{13}}{6} \sim 0,4342$. Then, we have to distinguish 2 cases.

- For all $a_1 \in (0, 1)$ given $a_2 < \tilde{a}_2$, δ is increasing for $a_1 \in (0, \underline{a}_1]$ and decreasing for $a_1 \in [\underline{a}_1, 1)$. It follows that $\Delta_1 + \Delta_2 > 0$.

- For all $a_1 \in [\tilde{a}_2, 1)$ such as $a_2 \in [\tilde{a}_2, \underline{a}_2]$, δ is increasing. Hence, $\Delta_1 + \Delta_2 > 0$.

Then, we have determined that the function δ is non-negative for all a_1 given each value of a_2 inferior to a_1. Hence, the result. ■

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