Runoff Elections and the Condorcet Loser: The Ortega Effect*

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This version:
December 30, 2008

Abstract

A crucial component of Runoff electoral systems is the threshold fraction of votes above which a candidate wins outright in the first round. I analyze the influence of this threshold on the voting equilibria in three-candidate Runoff elections. I demonstrate the existence of an *Ortega Effect* which may unduly favor dominated candidates and thus lead to the election of the Condorcet Loser in equilibrium. The reason is that, contrarily to commonly held beliefs, lowering the threshold for first-round victory may actually induce voters to express their preferences excessively. I also extend Duverger's Law to Runoff elections with any threshold below, equal or above 50%. Therefore, Runoff elections are plagued with inferior equilibria that induce either too high or too low expression of preferences.

JEL Classification: D72
Keywords: Runoff elections, Poisson Games, Condorcet Loser

*I thank Elena Arias Ortiz, Micael Castanheira, Paola Conconi, Frédéric Malherbe and seminar participants at ECARES for insightful comments. I acknowledge financial support from the *Fonds d'Encouragement à la Recherche de l'Université Libre de Bruxelles*. All remaining errors are mine.

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1 Introduction

Runoff electoral systems are widely used around the world and their popularity is rising. It is the single most used electoral system for presidential elections (Blais *et al* 1997) and many countries have adopted it over the last 30 years (Perez-Linan, 2006). In this electoral system, a candidate wins outright if he receives more than a pre-defined fraction of the votes. If no candidate passes this threshold, then a runoff is organized between the two candidates who received the largest number of first-round votes. This threshold for first-round victory is a crucial component of the Runoff electoral system. It varies widely across countries: beyond the "classic" 50%-threshold used for instance in France, many countries, states and cities have defined a threshold below 50%.[1] This paper focuses on the influence of this threshold on the properties of Runoff electoral systems.

Traditionally, countries define this threshold based on a simple trade-off between voters preference revelation and first-round decisiveness (Duverger 1963, Riker 1982, Cox 1997). It is indeed commonly believed that lowering the threshold increases the decisiveness of the first round and reduces voters revelation of preferences. The intuition is as follows. A lower threshold increases the probability that a candidate wins outright. This should therefore increase the voters' incentive to rally a "viable" candidate (i.e. a candidate that has a serious chance to win) in the first round even if this candidate is not their most preferred one. If true, this would reinforce the decisiveness of the first round because more votes would be concentrated on the top-two candidates. By contrast, a higher threshold decreases the probability that the first round directly selects a winner. Voters are thus less afraid of "wasting" their first-round ballot on a non-viable candidate. On the basis of this trade-off, it has for instance been argued that lower-than-50% thresholds avoid organizing useless second rounds (Shugart and Taagepera 1994, O'neil 2007).[2]

In this paper, I show that this perceived trade-off may not exist. Contrarily to these commonly held beliefs, lowering the threshold may actually induce voters to reveal their preferences excessively. Indeed, there always are cases in which a ballot has virtually no chance to determine who is the first-round winner. Then, rational voters will focus their attention towards determining who is the second candidate, in case a runoff is organized. They therefore fully reveal their preferences because they get trapped in a prisoners'

[1] Note that thresholds above 50% can also be chosen: in Sierra Leone, 55% of the votes are necessary to win in the first round.

[2] A second round is considered "useless" when the election winner corresponds to the winner of the first-round. This happens regularly. For instance, Bullock and Johnson (1992) report empirical evidence on US data according to which the election winner corresponds to the first-round winner approximately 70% of the times (see also Engstrom and Engstrom, 2008).

dilemma situation, where they overlook the decisiveness of the first round. As I show, this unduly favors otherwise dominated candidates.

To show this, I build a model of three-candidate Runoff elections in which a divided majority faces a unified minority.[3] All majority voters prefer either candidate A or B to a third candidate, C. They are divided as to which of A and B is the best candidate. The minority is instead unified behind candidate C. Since he would lose a one-to-one contest against every other candidate, candidate C is the Condorcet Loser. Using Poisson games (Myerson, 1998a, 1998b), I analyze the influence of the threshold level on equilibrium voting behavior. I show that, even though C is the least preferred candidate, he may systematically win the election if the threshold is below 50%, because of excessive vote dispersion. This excessive vote dispersion is what I call the *Ortega Effect*, named after Daniel Ortega, the winner of the 2006 Presidential election in Nicaragua (detailed below).

Why would majority voters divide their votes if they expect C to win? Consider the first-round choice of a majority-block voter who prefers B to A. If he expects that C will pass the threshold (which is below 50%) and win outright, his main objective is to prevent C's victory. The common belief is that, in such a situation, he will vote for the strongest candidate, say A, to beat C in the first round. If all majority voters do the same, candidate A indeed wins the election outright instead of C. Following this reasoning, the first round is "decisive" and the Condorcet Loser never wins in equilibrium. I show that this argument fails to take into account the threshold for first-round victory: the victory of C in the first-round can also be avoided simply by increasing the number of votes necessary to pass the threshold. Since the threshold is a percentage of the total number of votes, casting a ballot in favor of any majority candidate increases the number of votes necessary to pass the threshold. Thus, in the situation considered here, to prevent C's victory, the majority voter just has to vote for the candidate he prefers: B. Such a behavior explains why, when they expect C to win in the first round, majority voters may decide to disperse their votes. With a threshold below 50%, this excessive vote dispersion can be an equilibrium and may thus generate the systematic victory of the Condorcet Loser.

The puzzling result of the 2006 presidential election in Nicaragua (see Lean 2006) may be reinterpreted in light of this Ortega Effect. In this election, right-wing voters formed a majority of the electorate but were divided between two candidates: Eduardo Montealegre (ALN) and José Rizo (PLC). There was only a minority of left-wing voters, but they were staunchly supporting their sole "serious" contender: Daniel Ortega (FSNL).

[3] The issue of the divided majority is often considered in the literature on electoral systems. It is, for instance, at the heart of Borda's demonstration that Plurality may fail to aggregate preferences (Borda, 1781). See also Myerson and Weber (1993), Cox (1997), Piketty (2000), Myerson (2002), Martinelli (2002), and Dewan and Myatt (2007).

In other words, D. Ortega was the Condorcet Loser of this election.[4] Nicaragua's electoral system is a Runoff where a candidate wins outright if he obtains more than 40% of the votes or more than 35% of the vote and a victory margin over the nearest competitor of 5%. Before the election, polls indicated that, due to a division among right-wing voters, D. Ortega would win outright. Despite this information, right-wing voters divided their votes and D. Ortega won the presidential race with 38% of the votes (E. Montealegre and J. Rizo obtained, respectively, 28.3% and 27.1% of the votes). According to the traditional explanation, this result was an accident: right-wing voters should not have divided their votes. According to the *Ortega Effect* identified in this paper, it was not: ex ante it was individually rational for right-wing voters to divide their votes.

The Ortega Effect is a strong argument against Runoff systems with thresholds lower than 50%. Yet, the popularity of such low thresholds is rising (Shugart and Taagepera 1994, Shugart 2006, O'neil 2007): the threshold is, for instance, as low as 40% in Costa Rica, North Carolina and New York city. In Argentina and Ecuador, like in Nicaragua, the threshold depends on the margin of victory. The threshold is 45% (50% in Ecuador) but falls to 40% if a candidate has a 10-point lead over the nearest competitor. Note that the Ortega Effect extends to Runoff electoral systems with such victory margin requirements.

In this paper, I also show that Runoff electoral systems may induce insufficient dispersion of the votes. Indeed, even with a threshold equal or above 50%, first-round decisiveness may be "too high" for voters to reveal their preferences. In particular, I show that, unless the minority supporting C is really small, no matter the threshold (below, equal or above 50%), Runoff electoral systems always produce equilibria in which only two candidates obtain votes in the first round. In our three-candidate setup, this implies that some voters do not vote for their preferred candidate: they do not reveal their preferences in such an equilibrium. This is polar to the Ortega effect identified in this paper: voters want to avoid the victory of their least-preferred candidate, either in the first round (if the threshold is below 50%) or in the second round (if the threshold is equal or above 50%). This result actually contradicts Duverger's Hypothesis and extends Duverger's Law to Runoff elections (see Riker 1982). This is why I call these equilibria *Duvergerian*.[5]

To understand why Duvergerian equilibria exist in Runoff elections, consider again the choice of a majority-block voter who prefers B to A. This voter expects that, in the first round, all (or almost all) other majority-block voters vote for A and that voters of the minority-block vote for C. This means that candidate B's expected vote share is nil (or

[4] In fact, during the campaing, D. Ortega himself admitted that he was the Condorcet Loser.

[5] This result gives a theoretical justification for the empirical evidence according to which the number of effective candidates is below 3 in presidential Runoff elections around the world (Shugart and Taageepera 1994). Costa Rica illustrates the case of a Runoff election with only 2 serious parties.

extremely small). In such a situation, if the voter votes for A, he may prevent himself from the victory of C in the first round (if the threshold is below 50%) or from the risk of a victory of C in the second round (if the threshold is equal or above 50%). In comparison, voting for B is valuable if a second round is organized and if this ballot allows B to participate in that round. I prove that, in this equilibrium, unless the minority is really small, what matters for a majority-block voter is to avoid the risk of C's victory either in the first round or in the second round.[6] Consequently, all majority-block voters voting for one of the two majority candidates is an equilibrium in Runoff elections.

The paper is organized as follows: Section 2 lays out the setup. Section 3 details how voters decide for whom to vote. Section 4 analyzes equilibrium behavior under Runoff elections. Section 5 extends the results to different setups. Section 6 concludes.

2 Setup

There are three candidates, $P \in \{A, B, C\}$ and three types of voters, $t \in \{t_A, t_B, t_C\}$. The probability that a given voter has type t is denoted $r(t)$, with $\sum_t r(t) = 1$. I denote the utility of a type-t voter by the function $U(W|t)$, where W is the candidate winning the election. Thus, voters do not directly derive a benefit from the ballot they cast: they are instrumental.

Types t_C are called the *minority block*: in expected terms, they represent a minority of the electorate, i.e. $r(t_C) < 1/2$. They always prefer candidate C and, for expositional simplicity, I also assume that they are indifferent between the other two candidates:

$$\begin{aligned} U(W, t_C) &= 1 \text{ if } W = C \\ &= 0 \text{ if } W \in \{A, B\}. \end{aligned} \tag{1}$$

We will see that this implies that typest_C always vote for C (which greatly simplifies the analysis). I however prove that the main results are not affected by this assumption.

Together, types t_A and t_B are called the *majority block*: in expected terms, they represent a majority of the electorate, i.e. $r(t_A)+r(t_B) > 1/2$. They all identify candidate C as being the worst option but have different opinion about A and B. Types t_A prefer A to B whereas types t_B prefer B to A. Then, I have

$$\begin{aligned} U(W, t_A) &= 1 \text{ if } W = A \\ &= 0 \text{ if } W = B \\ &= -1 \text{ if } W = C \end{aligned} \tag{2}$$

[6] Taking into account the risk of upset victory explains the difference between my results and Martinelli (2002)'s results vis-à-vis Duvergerian equilibria in Runoff elections with a threshold equal to $\frac{1}{2}$.

for types t_A, and

$$U(W, t_B) = 1 \text{ if } W = B \qquad (3)$$
$$= 0 \text{ if } W = A$$
$$= -1 \text{ if } W = C$$

for types t_B.[7] To be sure that the results do not hinge on any form of symmetry, I assume that, in expected terms, types t_A represent a larger (or equal) fraction of the electorate than types t_B: $r(t_A) \geq r(t_B)$.

Runoff elections are held in one or two rounds. In the first round, each voter casts a ballot in favor of one of the competing candidates. The action set of the voters is denoted by $\Psi^1 = \{A, B, C\}$.[8] If the candidate that ranks first obtains more than a fraction ζ of the votes (called the threshold for first-round victory), he wins outright: there is no second round.[9] If no candidate passes this threshold, then a second round (a.k.a. *runoff*) is organized. In this second round, as in the first one, each voter casts a ballot in favor of one of the participating candidates. But, in this round, not all candidates participate: it opposes only the two candidates that received the most votes in the first round (called the top-two candidates). The action set of voters is denoted by $\Psi^2 = \{P, Q\}$, where P and Q refer respectively to the candidates that ranked first and second in the first round. If a second round is held, there is always a winner: the candidate that receives most votes in that last round. Note that ties are resolved by the toss of a fair coin.

In a three-candidate setup, a Runoff electoral system with a threshold below $\frac{1}{3}$ corresponds to the Plurality electoral system (a.k.a. *first-past-the-post)*. Indeed, since at least one candidate receives $\frac{1}{3}$ or more of the votes, a second round is never organized: the first round always determinates a winner. This is why I only consider Runoff electoral systems with a threshold $\zeta \geq \frac{1}{3}$.

I conduct the analysis under the assumption that the size of the electorate, k, is distributed according to a Poisson distribution of mean n (see, e.g. Myerson 2000 for the properties of Poisson games): $k \sim P(n)$. The probability that there are k voters in the electorate is therefore:

$$\Pr(k|n) = \frac{e^{-n} n^k}{k!}.$$

The number of players who choose action ψ in the first round is denoted by x^1_ψ, $\psi \in \Psi^1$. The number of players who choose action ψ in the second round is denoted by x^2_ψ, $\psi \in \Psi^2$.

[7] In Section 5.2, following Bouton and Castanheira (2008), I show that the results hold when majority voters are divided because they have opposite information instead of opposite preferences.

[8] Abstention is excluded from the action set but I will show that the results do not hinge on that assumption.

[9] In Section 5.1, I show that the results hold when first-round victory requires a victory margin.

We will see below that, given a Poisson-distributed total size of the population, each random variable x_{ψ}^{r}, with $r \in \{1,2\}$ itself follows a Poisson distribution. This implies that each voter has a strictly positive probability of affecting the outcome of the election.

A type t's *strategy function* is a mapping $\sigma^{r}(t) : t \to \psi$ that specifies a probability distribution over the set of actions Ψ^{r} for each type t. $\sigma^{r}(\psi|t)$ denotes the probability that a randomly sampled voter of type t plays action ψ in round r. The usual constraints apply: $\sigma^{r}(\psi|t) \geq 0$ and $\sum_{\psi} \sigma^{r}(\psi|t) = 1$, $\forall t$. Given the strategy function $\sigma^{r}(t)$ of each type t, a fraction:

$$\tau_{\psi}^{r} = \sum\nolimits_{t} r(t)\ \ \sigma^{r}(\psi|t) \tag{4}$$

of the electorate is expected to play action ψ in round r. We call τ_{ψ}^{r} the *expected share of voters* who choose action ψ in round r.

I analyze symmetric Bayesian Nash equilibria of this voting game for an expected population size n that becomes infinitely large. By the very nature of population uncertainty, the equilibrium mapping $\sigma^{r}(\psi|t)$ *must* be identical for all voters of a same type t (see Myerson 1998b, p377, for more detail).

3 Voting Behavior

Denoting by $\Pr(W)$ the probability that candidate $W \in \{A, B, C\}$ wins the election, the expected utility of a type-t voter is:

$$\mathsf{E}U(t) = \Pr(A)\,U(A|t) + \Pr(B)\,U(B|t) + \Pr(C)\,U(C|t) \tag{5}$$

Using (1), (2), and (3) the expected utility of the different types of voter are:

$$\mathsf{E}U(t_A) = \Pr(A) - \Pr(C) \tag{6}$$

for a type t_A,

$$\mathsf{E}U(t_B) = \Pr(B) - \Pr(C) \tag{7}$$

for a type t_B, and

$$\mathsf{E}U(t_C) = \Pr(C) \tag{8}$$

for a type t_C. Expression (6) means, for instance, that types t_A voting behavior aims at maximizing the probability that A wins and minimizing the probability that C wins.

The value of each action, and thus voters' behavior, depends on its probability of affecting the outcome of the election, *i.e.* on its probability of being *pivotal*. In Runoff elections, a ballot can be pivotal in the first and the second round. For subgame perfection reasons, I start by the analysis of the second-round pivotabilities and voting behavior, which are both straightforward.

In the second round, a ballot is pivotal when one candidate trails behind by exactly one vote or when the two candidates have the same number of votes. In a two-candidate election, like the second round, casting a ballot in favor of one candidate increases the probability of victory of this candidate and decreases the probability of victory of the other candidate. It immediately follows that:

Proposition 1 *In the second round subgame, voters always vote for the candidate they prefer. Thus, the expected results of the second round depends on the identity of the candidates participating to that round:*
(i) when $\Psi^2 = \{A, C\}$*:* $\tau_A^2 = r\,(t_A) + r\,(t_B) > \tau_C^2 = r(t_C)$*,*
(ii) when $\Psi^2 = \{B, C\}$*:* $\tau_B^2 = r\,(t_A) + r\,(t_B) > \tau_C^2 = r(t_C)$*,*
(iii) when $\Psi^2 = \{A, B\}$*:* $\tau_A^2 = r\,(t_A) + \sigma^2(A|t_C)r(t_C)$ *and* $\tau_B^2 = r(t_B) + \sigma^2(B|t_C)r(t_C)$*.*

When C participates to the second round, majority-block voters coordinate their votes on the participating majority candidate. Therefore, when opposed to C, both A and B win with a probability that tends to 1. When A and B are opposed, the result depends on the parameter values and types t_C strategies, i.e. $\sigma^2(A|t_C)$ and $\sigma^2(B|t_C)$. For the sake of simplicity, I assume that that types t_C mix equally between A and B if C is not among the top-two candidates, i.e. $\sigma^2(A|t_C) = \frac{1}{2} = \sigma^2(B|t_C)$.[10] Therefore, when opposed to B, A wins with a probability that tends to 1 except if $r\,(t_A) = r(t_B)$. In the latter case, A and B have an equal chance to win.

Knowing how voters behave in the second round, I can analyze what happens in the first round. In Runoff elections, the first round determines the final result either directly or indirectly. A ballot can directly affect the outcome of the election in two ways. First, if one candidate misses one vote to pass the threshold for first-round victory and if the other candidates are below the threshold. I then say that the ballot is *threshold pivotal*. The threshold pivotability is denoted by $piv^1_{i/ij}$, where the subscript i/ij means that candidate i misses one vote to pass the threshold and that without any other ballot in favor of i a second round opposing i to j will be organized. Second, if two candidates have (almost) the same number of votes and are both above the threshold, I say that the ballot is *above-threshold pivotal*. The above-threshold pivotability between i and j is denoted by $piv^1_{i/j}$.

If no candidate passes the threshold for first-round victory, the winner is indirectly decided in the first round through the identity of the candidates participating to the second

[10] Even if the outcome of the second round, and thus Proposition 1, are affected by this assumption, it is easy to show that the main results of the paper are not.

round (see Proposition 1). As explained above, in Runoff elections, the two candidates that participate to the second round are the candidates that rank first and second in the first round: the top-two candidates. Thus, a ballot changes the identity of the top-two candidates when it allows the candidate that ranks third to rank second. More precisely, I say that a ballot is *second-rank pivotal* when the candidates that rank second and third have the same number of votes or when the third trails behind by exactly one vote. The second-rank pivotability between i and j is denoted by piv^1_{ij}. Table 1 below summarizes the different first-round pivotal events that influence the first-round voting behavior.[11]

Table 1: first-round pivotal events.

Event	**Notation**	**Condition**
Threshold pivotal i/ij	$piv^1_{i/ij}$	$x^1_i = \zeta \left(x^1_i + x^1_j + x^1_k \right) > x^1_j > x^1_k$
Above-threshold pivotal i/j	$piv^1_{i/j}$	$x^1_i = x^1_j \geq \zeta \left(x^1_i + x^1_j + x^1_k \right) > x^1_k$
Second-rank pivotal ij	piv^1_{ij}	$x^1_i = x^1_j < x^1_k < \zeta \left(x^1_i + x^1_j + x^1_k \right)$

Let $G^1(\psi|t)$ denote the *expected gain* of playing action ψ in the first round, $\psi \in \Psi^1$. This gain depends on the voter's preference, summarized by $U(\cdot|t)$, and on the strategy function $\sigma^1 \equiv \left\{ \sigma^1(A|t), \sigma^1(B|t), \sigma^1(C|t) \right\}$ of the other voters. These strategies determine the expected number of votes received by each candidate in the first round, and thereby the pivot probabilities in the first round, i.e. $\Pr\left(piv^1_{i/ij}\right)$, $\Pr\left(piv^1_{i/j}\right)$ and $\Pr\left(piv^1_{ij}\right)$. For a type t, the expected gain of playing action i in the first round is:

$$\begin{aligned} G^1(i|t) = {} & \Pr\left(piv^1_{ij}\right)\left[U(k,i|t) - U(k,j|t)\right] + \Pr\left(piv^1_{ik}\right)\left[U(j,i|t) - U(j,k|t)\right] + \\ & \Pr\left(piv^1_{i/j}\right)\left[U(i|t) - U(j|t)\right] + \Pr\left(piv^1_{i/k}\right)\left[U(i|t) - U(k|t)\right] + \\ & \Pr\left(piv^1_{i/ik}\right)\left[U(i|t) - U(i,k|t)\right] + \Pr\left(piv^1_{i/ij}\right)\left[U(i|t) - U(i,j|t)\right] + \\ & \Pr\left(piv^1_{j/ji}\right)\left[U(j,i|t) - U(j|t)\right] + \Pr\left(piv^1_{j/jk}\right)\left[U(j,k|t) - U(j|t)\right] + \\ & \Pr\left(piv^1_{k/ki}\right)\left[U(k,i|t) - U(k|t)\right] + \Pr\left(piv^1_{k/kj}\right)\left[U(k,j|t) - U(k|t)\right], \end{aligned} \tag{9}$$

where $i, j, k \in \Psi^1$, $i \neq j \neq k$, and $U(i,j|t)$ denotes the expected utility for a type-t of a second round opposing i to j. This expected utility is defined as follows:

$$U(i,j|t) = \Pr(i|\{i,j\})\, U(i|t) + \Pr(j|\{i,j\})\, U(j|t) \tag{10}$$

where $\Pr(i|\{i,j\})$ is the probability that candidate i wins the second round if opposed to candidate j.

[11] Note that I do not omit three-way ties in the analysis. There are just considered as a specific case of two-way ties.

The first line in (9) reads as follows: if a ballot in favor of i is second-rank pivotal ij, then the second round opposes k to i instead of k to j; if a ballot is second-rank pivotal ik, then the second round opposes j to i instead of j to k. The second line concerns the gains when the ballot is above-threshold pivotal and the three last lines concern the gains when the ballot is threshold pivotal.

Comparing the expected gain of playing the different actions, the first-round voting behavior of types t_C is straightforward: they always vote for C. Technically, $G^1(C|t_C) > G^1(A|t_C)$ and $G^1(C|t_C) > G^1(B|t_C)$. This is so because, contrarily to A and B ballots, a C-ballot increases the probability of C's victory no matter the situation.[12] Since types t_C are indifferent between A and B (from (1)), their only objective is to maximized the probability of C's victory. They therefore prefer to cast a ballot in favor of C. Putting this result together with Proposition 1, I have that types t_C vote for candidate C in both rounds. From hereon, I therefore focus our attention on the voting behavior of types t_A and t_B.

To analyze the voting behavior of types t_A and t_B, it is necessary to have more details about the first-round pivot probabilities: $\Pr\left(piv^1_{i/ij}\right)$, $\Pr\left(piv^1_{i/j}\right)$ and $\Pr\left(piv^1_{ij}\right)$. These pivot probabilities depend on the distribution of the number x^1_ψ of voters who play each action $\psi \in \Psi^1$. As shown by Myerson (1998a, 1998b, 2000), since the total number of voters follows a Poisson distribution of mean n, the realizations x^1_ψ follow mutually independent Poisson distributions: $x^1_\psi \sim \mathcal{P}\left(n \cdot \tau^1_\psi\right)$, where τ^1_ψ is the expected fraction of voters playing action ψ in the first round (see (4) above). Property 1 below summarizes some of the properties proven by Myerson (1998a, 1998b, 2000).

Property 1 *(Myerson 2000, Theorem 1 and Corollary 1). Subject to $\sum_{\psi \in \{A,B,C\}} \tau^1_\psi = 1$, given the expected numbers of votes $n\tau^1$, the probability that the realized number of votes are $x^1 = \left\{x^1_A, x^1_B, x^1_C\right\}$ is:*

$$\Pr(x^1|\tau^1) \underset{n^1 \longrightarrow \infty}{\longrightarrow} \max_{x^1} \frac{\exp[\sum_\psi \frac{x^1_\psi}{n}\left(1 - \log(\frac{x^1_\psi}{n\tau^1_\psi})\right) - 1]}{\prod\limits_{\psi \in \Psi} \sqrt{2\pi x^1_\psi + \frac{\pi}{3}}}, \tag{11}$$

For an increasingly large electorate size ($n \to \infty$), the probability of an event $x^1 =$

[12] There are two situations in which casting a ballot in favor of one of the majority candidate can increase the probability of C's victory. In these cases, a ballot in favor of C increases this probability in the same manner. Indeed, a A-ballot (resp. B-ballot) increases the probability of C's victory when the ballot is threshold pivotal B/BC (resp. A/AC) because it prevent candidate B (resp. A) to pass the threshold. But, if threshold pivotal B/BC (resp.A/AC), a ballot in favor of C also prevent such a first-round victory. Therefore, although A and B ballots can increase the probability of C's victory, types-t_C always prefer to vote for C.

$\{x_A^1, x_B^1, x_C^1\}$ *converges to zero at an exponential rate called the* magnitude *of the probability. We denote it mag* (x^1)*:*

$$mag\left(x^1\right) \equiv \lim_{n\to\infty} \frac{\log\left[\Pr\left(x^1\right)\right]}{n} \quad (\in [-1,0])$$
$$= \sum_{\psi} \frac{x_\psi^1}{n}\left(1 - \log\left(\frac{x_\psi^1}{n\tau_\psi^1}\right)\right) - 1. \qquad (12)$$

If two events have a different magnitude, then:

$$\lim_{n\to\infty} \frac{\Pr\left(x^1\right)}{\Pr\left(x^{1\prime}\right)} = 0 \text{ if and only if } mag\left(x^1\right) < mag\left(x^{1\prime}\right), \qquad (13)$$

with $x^1 \neq x^{1\prime}$.

The result summarized by equations (12) and (13) has been called the *magnitude theorem* by Myerson (2000). The intuition is that the probabilities of different events do not converge towards zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity.

These properties are quite general and not specific to Poisson games. For instance, Myerson (2000, Section 4) shows that pivot probabilities under multinomial distributions are simply a monotone transformation of their Poisson equivalent.[13] This is why my results extend directly to the multinomial distribution.

Property 2 in Appendix A1 uses these properties of Poisson games to compute the magnitude of the first-round pivot probabilities in Runoff elections.

4 Equilibrium Analysis

In this section, I prove two equilibrium properties of Runoff elections. First, I show that lowering the threshold for first-round victory below 50% may induce voters to reveal their preferences excessively. In particular, when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, the Condorcet Loser, C, may win the election in the first round because all majority vote for the candidate they prefer instead of coordinating their votes on one of the majority candidates. As explained in the introduction, I call this excessive dispersion of votes the *Ortega Effect* in reference of Daniel Ortega, the winner of the 2006 presidential elections in Nicaragua. Second, I show that Runoff electoral systems do not ensure that voters reveal their preferences more than

[13] Myerson (2000) shows that limits of pivot probabilities under Poisson games are such that $\lim_{n\to\infty} \log\left(\Pr\left(piv_{PQ}\right)\right)/n = \mu$. In his Section 4, Myerson (2000) shows that, if the distribution is Multinomial instead of Poisson, then $\lim_{n\to\infty} \log\left(\Pr\left(piv_{PQ}\right)\right)/n = \log(\mu + 1)$, where μ is the limit under the Poisson distribution. Therefore, the limit likelihood ratio (13) is the same under both distributions.

in Plurality. To do so, I extend the famous Duverger's Law (Duverger 1963), that says that there are only two "viable" candidates in Plurality elections, to Runoff elections. In particular, I prove that, unless $r(t_C)$ is too small, Runoff electoral systems always produce equilibria in which only two candidates receive votes. This *Duvergerian equilibria* are shown to exist for all $\zeta \in \left[\frac{1}{3}, 1\right)$.

4.1 The Ortega Effect

Knowing that, if C participates to the second round majority-block voters coordinate on the majority candidate (Proposition 1) and that $r(t_C) < 1/2$, it is impossible that candidate C wins in the second round with a probability that tends to 1. We thus focus our attention on the first round. In that round, C is elected with a probability that tends to 1 if the expected vote share of candidate C is both above the threshold and above the expected vote shares of candidates A and B:

$$\tau_C^1 > \zeta, \tag{14}$$

$$\tau_C^1 > \max[\tau_A^1, \tau_B^1]. \tag{15}$$

From $r(t_C) < 1/2$, I have that $\tau_C^1 > \max[\tau_A^1, \tau_B^1]$ is possible if and only if majority-block voters divide their votes, i.e. $\sigma^1(A|t_A) > 0$ and $\sigma^1(B|t_B) > 0$. In equilibrium,

$$\begin{aligned} G^1(A|t_A) - G^1(B|t_A) &\geq 0, \\ G^1(A|t_B) - G^1(B|t_B) &\leq 0, \end{aligned} \tag{16}$$

must then be satisfied. From (9) and Property 1, a sufficient condition for (16) to be strictly satisfied is that

$$\begin{aligned} mag\left(piv_{AB}^1\right) > \max\{ & mag\left(piv_{AC}^1\right), mag\left(piv_{BC}^1\right), mag\left(piv_{A/B}^1\right), \\ & mag\left(piv_{A/C}^1\right), mag\left(piv_{B/C}^1\right), mag\left(piv_{A/AB}^1\right), \\ & mag\left(piv_{B/BA}^1\right), mag\left(piv_{A/AC}^1\right), mag\left(piv_{B/BC}^1\right)\}. \end{aligned} \tag{17}$$

Indeed, in such a situation I have that:

$$\begin{aligned} G^1(A|t_A) - G^1(B|t_A) &\simeq U(C, A|t_A) - U(C, B|t_A) > 0, \\ G^1(A|t_B) - G^1(B|t_B) &\simeq U(C, A|t_B) - U(C, B|t_B) < 0. \end{aligned}$$

The intuition is as follows. When (17) is satisfied, majority-voters realize that, if pivotal, their ballot changes the identity of the majority candidate qualified for the second round, i.e. A is opposed to C if they vote for A and B is opposed to C if they vote for B.

Since, in the second round, all majority voters coordinate on the majority candidate that opposes C, this majority candidate is almost sure to win (see Proposition 1). Knowing this, typest_A strictly prefer to vote for A and typest_B strictly prefer to vote for B.

To prove that C may be elected in equilibrium, it is therefore sufficient to prove that (17) may be true when (14) and (15) are satisfied. This is what I do to prove the following Theorem:

Theorem 1 *For any Runoff elections with a threshold below* 50%*, there are distributions of preferences in the electorate such that, for the following equilibrium strategies*

$$\sigma^{1*}(A|t_A) = \sigma^{1*}(B|t_B) = \sigma^{1*}(C|t_C) = 1,$$

C *wins outright in the first round. This is what I call the* Ortega Effect*.*

Proof. See Appendix A2. ■

Why would majority voters divide their votes when C is expected to win? Consider the first-round choice of a majority-block voter who prefer B to A. If he expects that C will pass the threshold (which is below 50%) and then win outright, his main objective is to prevent C's victory. The common belief is that, in such a situation, majority voters vote for the strongest majority candidate, say A, to beat C directly. Candidate A would then win the election in the first round instead of C. Following this reasoning, majority voters divide their votes only when they fail to decide on which candidate to coordinate, i.e. if there is no strongest majority candidate. The Condorcet Loser can thus not win systematically.

This argument does not take into account that the victory of C in the first-round may also be avoided by increasing the number of votes necessary to pass the threshold. Indeed, if there is a second round, all majority voters coordinate on the remaining majority candidate which ensures his victory. To increase the number of votes necessary to pass the threshold, majority voters have to cast a ballot in favor of one the majority candidate, no matter which one. Indeed, the threshold is a percentage of the total number of votes. Thus, to prevent C's victory, it may be individually rational for majority voters to vote for the candidate they prefer in order to make him win through a qualification for the second round. In Theorem 1 I prove that, for any first-round victory threshold below 50%, there are distribution of preferences in the electorate such that, in equilibrium, C wins outright because all voters vote for their preferred candidate.

Let me illustrate the result in Theorem 1 through a numerical example. Suppose $\zeta = 0.4$, $r(t_A) = 0.30$, $r(t_B) = 0.29$, and $r(t_C) = 0.41$. With these parameter values, as for the Nicaraguan case discussed in the introduction, the Condorcet Loser, C, would asymptotically be sure to win the election in the first round if the majority divide their votes. First-round vote shares would indeed be: $\tau^1(C) = 0.41 > \zeta > \tau^1(A) = 0.30 > \tau^1(B) = 0.29$. Even though, for these parameter values, $\sigma^1(A|t_A) = \sigma^1(B|t_B) = \sigma^1(C|t_C) = 1$ is an equilibrium strategy profile since, as illustrated in Table 1, (17) is satisfied:

Table 2: equilibrium magnitudes.

Threshold mag.*	**Above-Threshold mag.****	**Second-rank mag.*****
$mag(piv^1_{A/AC}) = -0.0223$	$mag(piv^1_{A/C}) = -0.0304$	$mag(piv^1_{AC}) = -0.0125$
$mag(piv^1_{B/BC}) = -0.0273$	$mag(piv^1_{A/B}) = -0.0953$	$\mathbf{mag(piv^1_{AB}) = -0.0003}$
$mag(piv^1_{A/AB}) = -0.0310$	$mag(piv^1_{B/C}) = -0.0370$	$mag(piv^1_{BC}) = -0.0125$
$mag(piv^1_{B/BA}) = -0.0343$		
$\mathbf{mag(piv^1_{C/CA}) = -0.0002}$		
$mag(piv^1_{C/CB}) = -0.0003$		

* Threshold pivotal $(piv^1_{i/ij})$ if $x^1_i = \zeta\left(x^1_i + x^1_j + x^1_k\right) > x^1_j > x^1_k$

** Above-threshold pivotal $(piv^1_{i/j})$ if $x^1_i = x^1_j \geq \zeta\left(x^1_i + x^1_j + x^1_k\right) > x^1_k$

*** Second-rank pivotal (piv^1_{ij}) if $x^1_i = x^1_j < x^1_k < \zeta\left(x^1_i + x^1_j + x^1_k\right)$

Because of $mag(piv^1_{C/CA})$, majority-block voters vote for one of the majority candidates to increase the first-round victory threshold and, because of $mag(piv^1_{AB})$, they choose the candidate they prefer among the majority candidates, i.e. candidate A for typest_A and candidate B for typest_B.

This numerical example also helps to understand why the result in Theorem 1 holds when (i) types t_C have strict preferences over A and B, and when (ii) abstention is included in the action set. Relaxing these two assumptions does not influence the Ortega Effect because types t_C wants to ensure C's victory in the first round. Therefore they do not want to abstain or to vote for A or B. Technically, types t_C strictly prefer to cast a ballot in favor of C because $mag(piv^1_{C/CA})$ is the largest magnitudes.

4.2 Duvergerian Equilibria

In Runoff elections, the reason that explains the existence of Duvergerian equilibria, i.e. equilibria in which only two candidates receive vote, is the following. Consider the first-round strategy profile $\sigma^1(B|t_B) = 1$ and $\sigma^1(B|t_A) \to 1$, for which alternative A's expected vote share is vanishingly small. What is a given t_A-voter's best response? If he plays

$\psi = B$ and is pivotal to elect B in the first round, he saves himself either from a victory of C in the first round when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$ (if above-threshold pivotal B/C) or from the risk of an upset victory of C in the second round when $\zeta \in \left[\frac{1}{2}, 1\right)$ (if threshold pivotal B/BC). In comparison, action $\psi = A$ is valuable if a second round is organized and if his ballot is pivotal in bringing A to that round (if second-rank pivotal AC). Comparing the probabilities of each of these events shows that:

Theorem 2 *In Runoff elections, there always exist two equilibria in which all majority types play $\psi = A$ (resp. B). When $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, these equilibria exist for any $0 < r(t_C) < 1/2$. When $\zeta \in \left[\frac{1}{2}, 1\right)$, these equilibria exist for any $0.06699 < r(t_C) < 1/2$.*

Proof. See Appendix A2. ■

The trade-off is self-explanatory. A majority voter has an incentive to abandon a trailing candidate (A in the above case) if the risk of a victory of C (either in the first or in the second round) is too high compared to the first-round chances of bringing the trailing candidate to the second round. Typically, the larger C's vote share, the higher the risk of C's victory, and the lower the probability that one vote may bring A to the second round.

5 Extensions

In this section, I extend the setup in two directions. First, I analyze Runoff electoral systems that impose an extra condition for first-round victory: a victory margin requirement. In these electoral systems, a candidate wins in the first round if he receives more than a fraction ζ of the votes *and* if he has a β-points lead over the nearest competitor. Such a Runoff election is, for instance, used in Argentina, Ecuador and Nicaragua. In the first subsection, I prove that the Ortega Effect holds for Runoff elections that impose victory margin requirements. Second, I consider a different source of majority divisions: majority voters are divided because of information instead of preferences. In Bouton and Castanheira (2008), it is shown that the nature of majority divisions may dramatically affect the properties of electoral systems. Traditionally, it is assumed that majority voters are divided because they divergent preferences over candidates (as in our setup). Bouton and Castanheira (2008) show that if, instead, majority is divided because voters hold different information about the candidates, then electoral systems properties may be different. In the second subsection, I prove that the Ortega effect holds when majority divisions result from divergent information instead of divergent preferences.

5.1 Victory Margin

Imposing a victory margin requirement has two consequences. First, there is an additional condition for a ballot to be threshold pivotal. Without a victory margin requirement, a ballot is threshold pivotal i/ij when candidate i receives exactly a fraction ζ of the votes and the ranking is $i > j > k$, i.e. $x_i^1 = \zeta\left(x_i^1 + x_j^1 + x_k^1\right)$ and $x_i^1 > x_j^1 > x_k^1$. Now, in addition to these conditions, candidate i must have a lead over the other competitors larger or equal to a fraction β of the votes, i.e. $x_i^1 - x_j^1 > \beta\left(x_i^1 + x_j^1 + x_k^1\right)$ and $x_i^1 - x_k^1 > \beta\left(x_i^1 + x_j^1 + x_k^1\right)$. Since the magnitude of the threshold pivot probability can only be affected negatively by a new constraint, I have that the threshold-pivot probability when a victory margin is required, denoted $mag(piv_{i/ij}^{1,VM})$, is smaller or equal to the one without the additional constraint, i.e.

$$mag(piv_{i/ij}^{1,VM}) \leq mag(piv_{i/ij}^{1}), \tag{18}$$

where $mag(piv_{i/ij}^{1})$ is as defined in Property 2.

The second consequences is that there is, in Runoff electoral systems with victory margin requirement, a new type of first-round pivotability. Indeed, a ballot can change the outcome of the election when it allows one candidate to have a β-point lead over his nearest competitor. In such a situation, I say that a ballot is *margin pivotal* $i-j$, denoted $piv_{i-j}^{1,VM}$. More precisely, this happens when

$$\begin{aligned} x_i^1 - x_j^1 &= \beta\left(x_i^1 + x_j^1 + x_k^1\right), \\ x_i^1 &> \zeta\left(x_i^1 + x_j^1 + x_k^1\right), \text{ and} \\ x_j^1 &> x_k^1. \end{aligned}$$

To prove that the Ortega effect holds in Runoff elections with victory margin requirements, it is sufficient to prove that among the pivotabilities that influence the first-round voting behavior of types t_A and t_B, $\Pr\left(piv_{AB}^1\right)$ still has the largest magnitude when $\tau_A^1 = \tau_B^1 < \tau_C^1$ and $\tau_C^1 \to \zeta$. Since $mag(piv_{AB}^1) \to 0$ for these expected vote shares, I have to prove that

$$\begin{aligned} mag(piv_{i/ij}^{1,VM}) &< 0, && (19) \\ mag(piv_{i-j}^{1,VM}) &< 0, && (20) \end{aligned}$$

when $\tau_A^1 = \tau_B^1 < \tau_C^1$ and $\tau_C^1 \to \zeta$.

Knowing that (see proof of Theorem 1)

$$mag(piv_{i/ij}^{1}) < 0, \ \forall i,j \in \Psi^1,$$

from (18) I have that $mag(piv_{i/ij}^{1,VM}) < 0$.

From Property 1, I have that

$$\begin{aligned} mag(piv_{i-j}^{1,VM}) &= 0 \text{ if } \begin{cases} \tau_i^1 - \tau_j^1 = \beta \\ \tau_i^1 > \zeta \\ \tau_j^1 \geq \tau_k^1 \end{cases} \\ mag(piv_{i-j}^{1,VM}) &< 0 \text{ otherwise.} \end{aligned} \tag{21}$$

Thus, except for the non-generic cases for which the conditions in (21) are satisfied when $\tau_A^1 = \tau_B^1 < \tau_C^1$ and $\tau_C^1 \to \zeta$, I have that $mag(piv_{i-j}^{1,VM}) < 0$ when $\tau_A^1 = \tau_B^1 < \tau_C^1$ and $\tau_C^1 \to \zeta$.

I have then proven that both (19) and (20) are satisfied when $\tau_A^1 = \tau_B^1 < \tau_C^1$ and $\tau_C^1 \to \zeta$. Therefore, the Ortega Effect holds for Runoff elections in which there is a victory margin requirement for first-round victory.

A particular version of the victory margin requirements, called the double complement rule, has been proposed by Shugart and Taagepera (1994) for determining the situations for which a second round is necessary. Under their rule, if $x_i^1 > x_j^1 > x_k^1$, a second round is necessary if $2\left(\frac{1}{2} - \frac{x_j^1}{(x_i^1+x_j^1+x_k^1)}\right) > \frac{1}{2} - \frac{x_i^1}{(x_i^1+x_j^1+x_k^1)}$. Their derivation of this rule is simple: this is the equivalent of the arithmetic average of the rule for choosing a plurality winner, i.e. $x_i^1 > x_j^1$, and the rule for choosing a majority winner, i.e. $x_i^1 > \frac{1}{2}\left(x_i^1 + x_j^1 + x_k^1\right)$. The double complement rule is considered as a good compromise between Plurality and Runoff: a way to solve the trade-off between preferences revelation and first-round decisiveness explained in introduction. Testing this rule directly on actual Runoff elections data (Shugart and Taagepera, 1994; O'neil, 2007) leads to the following conclusion: the double complement rule prevents most unnecessary second rounds, i.e. second rounds in which the winner is the same as the first-round winner. Nonetheless, such a test does not take account of the influence of institutions on voters' incentives. The Ortega effect shed a new light on the effect that the double complement rule may have on voters' incentives and strongly qualifies the urge for a lowering of the thresholds for first round victory; even when victory margin requirements are imposed.

5.2 Nature of Majority Divisions

To prove that the Ortega effect holds when majority divisions result from divergent information instead of divergent preferences, I introduce state-contingent preferences as in Bouton and Castanheira (2008): which of the two majority candidates is best depends on the actual state of nature. Majority divisions result from voters having opposite beliefs about which state of nature prevails. Thus, I consider the same setup as before except for the preferences of the majority voters. There is now two states of nature: $\omega \in \{a, b\}$ and, conditional on the state of nature, types t_A and t_B hold identical preferences: they always want to elect the best candidate, which is A in state a and B in state b:

$$\begin{aligned} U(W, t_A, \omega) = U(W, t_B, \omega) &= 1 \text{ if } (W, \omega) = (A, a) \text{ or } (B, b) \\ &= 0 \text{ if } (W, \omega) = (A, b) \text{ or } (B, a) \\ &= -1 \text{ if } W = C, \end{aligned} \tag{22}$$

where $U(W, t, \omega)$ denotes the utility of a voter with type t when candidate W is elected and the true state is ω.

Each state of nature occurs with probability 1/2. The difference between types t_A and t_B stems from the fact that $r(t_A|a) > r(t_A|b)$. That is, there are more voters with type t_A in state a than in state b. The only information available to voters is their type. Knowing their type, they form beliefs about the probability of each state of nature. By Bayesian updating, a type-t voter infers that:

$$q(\omega|t) = \frac{r(t|\omega)}{r(t|a) + r(t|b)}.$$

Since $r(t_A|a) > r(t_A|b)$, I have:

$$q(a|t_A) > \frac{1}{2} > q(b|t_A) \text{ and } q(b|t_B) > \frac{1}{2} > q(a|t_B). \tag{23}$$

We still assume that, in expected terms, types t_A represent a larger (or equal) fraction of the electorate than types t_B:

$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b) \tag{24}$$

Note that the fraction of type t_C does not vary with the state of nature, i.e. $r(t_C|a) = r(t_C|b) = r(t_C)$.

Finally, the expected share of voters who choose action ψ in state ω in round r is

$$\tau_\psi^r(\omega) = \sum\nolimits_t r(t|\omega) \ \sigma^r(\psi|t).$$

From (22), I have that when C participates to the second round, voters always vote for the candidate they prefer: majority-block voters vote for the majority candidate and types t_C vote for candidate C. Knowing this, it is sure that C cannot be elected in the second round with a probability that tends to 1. To show that the Ortega effect holds when majority divisions result from voters having opposite beliefs vis-à-vis the actual state of nature, I then need to check whether $\sigma^1(A|t_A) = \sigma^1(B|t_B) = \sigma^1(C|t_C) = 1$ may be equilibrium strategies when

$$\tau_C^1(\omega) > \zeta, \tag{25}$$

$$\tau_C^1(\omega) > \max[\tau_A^1(\omega), \tau_B^1(\omega)], \ \forall \omega. \tag{26}$$

From $r(t_C) < 1/2$, I have that (26) is possible if and only if majority-block voters divide their votes: $\sigma^1(A|t_A) > 0$ and $\sigma^1(B|t_B) > 0$. In equilibrium I must then have

$$\begin{aligned} G^{1,I}(A|t_A) - G^{1,I}(B|t_A) &\geq 0, \\ G^{1,I}(A|t_B) - G^{1,I}(B|t_B) &\leq 0, \end{aligned}$$

where $G^{1,I}(\psi|t)$ is the expected gain of playing action ψ for a type-t in the first round. From (22), the expected gain of playing action i for a type-t in the first round is:

$$\begin{aligned} G^{1,I}(i|t) = &\sum_{\omega\in\{a,b\}} q(\omega|t)\{U(i,j|t,\omega)[\Pr(piv^1_{ik}|\omega) + \Pr\left(piv^1_{j/ji}|\omega\right) - \Pr\left(piv^1_{i/ij}|\omega\right)] + \\ &U(i,k|t,\omega)[\Pr(piv^1_{ij}|\omega) + \Pr\left(piv^1_{k/ki}|\omega\right) - \Pr\left(piv^1_{i/ik}|\omega\right)] + \\ &U(j,k|t,\omega)[\Pr\left(piv^1_{j/jk}|\omega\right) + \Pr\left(piv^1_{k/kj}|\omega\right) - \Pr(piv^1_{ij}|\omega) - \Pr(piv^1_{ik}|\omega)] + \\ &U(i|t,\omega)[\Pr\left(piv^1_{i/j}|\omega\right) + \Pr\left(piv^1_{i/k}|\omega\right) + \Pr\left(piv^1_{i/ik}|\omega\right) + \Pr\left(piv^1_{i/ij}|\omega\right)] - \\ &U(j|t,\omega)[\Pr\left(piv^1_{i/j}|\omega\right) + \Pr\left(piv^1_{j/ji}|\omega\right) + \Pr\left(piv^1_{j/jk}|\omega\right)] - \\ &U(k|t,\omega)[\Pr\left(piv^1_{i/k}|\omega\right) + \Pr\left(piv^1_{k/ki}|\omega\right) + \Pr\left(piv^1_{k/kj}|\omega\right)]\}. \end{aligned}$$

Note that the magnitudes are similar to those defined in Property 2. The only difference is that $\tau^1_\psi(\omega)$ have to be used to compute the magnitudes in state ω.

Knowing the expected gain of the different actions, a sufficient condition to have

$$G^{1,I}(A|t_A) - G^{1,I}(B|t_A) > 0 \text{ and } G^{1,I}(A|t_B) - G^{1,I}(B|t_B) < 0,$$

is that $mag(piv^{1,I}_{AB}|a) = mag(piv^{1,I}_{AB}|b)$ and that these magnitudes are the largest. Indeed, if this is satisfied, I have that

$$\begin{aligned} G^{1,I}(A|t_A) - G^{1,I}(B|t_A) &\simeq 2q(a|t_A)\Pr(A|\{C,A\}) - \\ &\quad 2q(b|t_A)\Pr(B|\{C,B\}) > 0, \qquad (27) \\ G^{1,I}(A|t_B) - G^{1,I}(B|t_B) &\simeq 2q(a|t_B)\Pr(A|\{C,A\}) - \\ &\quad 2q(b|t_B)\Pr(B|\{C,B\}) < 0. \qquad (28) \end{aligned}$$

The inequality signs in (27) and (28) come from (23) and from $\Pr(A|\{C,A\}) = \Pr(B|\{C,B\})$.

To prove that C may be elected in equilibrium, it is therefore sufficient to prove that $mag(piv^{1,I}_{AB}|a) = mag(piv^{1,I}_{AB}|b)$ may be the largest magnitude when (14) and (15) are satisfied. Knowing the proof of Theorem 1, this is straightforward. Indeed, from Property 2, I have that $mag(piv^{1,I}_{AB}|a) = mag(piv^{1,I}_{AB}|b)$ when

$$\tau^1_A(a) = \tau^1_B(b) \text{ and } \tau^1_A(b) = \tau^1_B(a).$$

Therefore, if

$$\begin{aligned} r(t_A|a) &= r(t_B|b), \text{ and} \\ r(t_A|b) &= r(t_B|a), \end{aligned}$$

I have that $mag(piv_{AB}^{1,I}|a) = mag(piv_{AB}^{1,I}|b)$ when $\sigma^1(A|t_A) = \sigma^1(B|t_B) = \sigma^1(C|t_C) = 1$. Under this assumption, the proof of Theorem 1 can be applied directly.

Finally, since all magnitudes are continuous in $r(t_A|\omega)$ and $r(t_B|\omega)$, there always exist other parameter values such $\sigma(A|t_A) = \sigma(B|t_B) = \sigma(C|t_C) = 1$ are equilibrium strategies for which C is elected in the first round.

6 Conclusion

This paper analyzed the influence of the threshold for first-round victory on voting equilibria in three-candidate Runoff elections. I modeled Runoff elections in which a divided majority faces a unified minority and I demonstrated the existence of two kind of equilibria: one highlights the *Ortega Effect*, the other extends *Duverger's Law* to Runoff elections. The Ortega Effect shows that, contrarily to commonly held beliefs, lowering the threshold for first-round victory may actually induce voters to reveal their preferences excessively. This excessive dispersion of vote may lead to the election of the Condorcet Loser in equilibrium. Vis-a-vis Duverger's Law, I showed that, no matter the threshold, Runoff electoral systems cannot ensure that voters reveal their preferences. Indeed, even with a threshold equal or above 50%, there always exist equilibria (called Duvergerian) in which some voters do not vote for their most preferred candidate: first-round decisiveness may be too high for all voters to reveal their preferences.

The Ortega Effect arises when it is very unlikely that a majority voter's ballot prevents an outright victory in the first round. In such a situation, majority voters focus on determining who is the second candidate participating to the runoff, even though it is very unlikely that such a runoff is organized. They thus vote for their most preferred candidate and not for their most preferred *serious* candidate. Doing so, they overlook the decisiveness of the first round and allow for the victory of the Condorcet Loser in equilibrium.[14] Duvergerian equilibria always exist because majority voters fear C's victory both in the first *and* in the second round. Indeed, when C is a threat, a candidate that has no serious chance to win the election may be abandoned by his supporters. Some voters may therefore not reveal their preferences in equilibrium.

[14] This gives a rational for the puzzling result of the 2006 Presidential election in Nicaragua. During this election, the Condorcet Loser (Daniel Ortega) won because majority voters divided their votes excessively. They did so even though polls predicted Ortega's victory due to divisions in the majority.

These results are a strong argument against Runoff electoral systems. Due to the Ortega Effect, this is particularly salient for Runoff electoral systems with thresholds below 50% as in some Latin American countries (Argentina, Nicaragua, Costa Rica, etc.) and US states and cities (North Carolina, New York, etc.). The Ortega Effect also sheds a new light on the literature that urges a lowering of the thresholds for first-round victory (Shugart and Taagepera, 1994; Shugart, 2006; O'neil, 2007).

Even if the model focused on a stylized case, the insights seem quite general. First, the Ortega Effect proved robust to two extensions. Indeed, the it holds in Runoff electoral systems that impose first-round victory margin requirements, and it also holds if majority voters are divided because of information instead of preferences (Bouton and Castanheira 2008). Second, the Ortega Effect should also be robust to extensions not formally considered in this paper. For instance, even if voters had different preference intensities, the Ortega Effect should still exist. Indeed, in the equilibrium sustaining the Ortega Effect, all majority voters vote for the candidate they prefer. Preferring a candidate more intensely cannot affect such strategies nor the outcome they imply. Similarly, the Ortega Effect should hold for a larger number of candidate. The three-candidate setup considered in this paper indeed focuses on the three candidates "viable" in Runoff elections.

According to the results in this paper, an optimal Runoff electoral system does not exist. Indeed, to get rid of both the Ortega Effect and Duvergerian equilibria the only way is to hinder fully the possibility to win the election outright in the first round. In other words, the threshold for first-round victory must be fixed at 100%. Such a Runoff system would nonetheless be extremely costly: when there are n candidates, $n-1$ rounds have to be organized. Any electoral reform should then consider other electoral systems like Approval Voting, Instant Runoff and Storable Votes. However, it is crucial to analyze and compare the equilibrium properties of these systems before any reform.

References

[1] Blais, André, Laslier Jean-François, Laurent, Annie, Sauger, Nicolas, and Van der Straeten, Karine (2007). "One Round versus Two Round Elections: An Experimental Study." French Politics 5 (3), pp. 278-286.

[2] Blais, André, Massicotte, Louis, and Dobrzynska, Agnieszka (1997). "Direct Presidential Elections: A Wolrd Summary." *Electoral Studies* 16 (4), pp. 441-455.

[3] Bouton, Laurent and Castanheira, Micael (2008). "One Person, Many Votes: Divided Majority and Information Aggregation." CEPR Discussion Paper, 6695.

[4] Bouton, Laurent and Castanheira, Micael (forthcoming). "The Condorcet-Duverger Trade-Off: Swing Voters and Electoral Systems." in Aragonés, E., Beviá, C., Llavador, H., Norman, S., Eds., *The Political Economy of Democracy*, fundacion BBVA, Spain

[5] Cox, Gary (1997). *Making Votes Count.* Cambridge, UK: Cambridge University Press.

[6] Dewan, Torun and Myatt, David (2007). "Leading the Party: Coordination, Direction, and Communication." *American Political Science Review*, Vol. 101, No. 4, pp. 825-843.

[7] Duverger, Maurice (1954). *Political Parties.* New York: John Wiley & Sons.

[8] Lean, Sharon (2007). "The Presidential and Parliamentary elections in Nicaragua, November 2006." Electoral Studies 26 (4), pp. 828-832.

[9] Martinelli, Cesar (2002). "Simple Plurality Versus Plurality Runoff with Privately Informed Voters." *Social Choice and Welfare* 19, pp. 901-919.

[10] Myerson, Roger (1998a). "Extended Poisson Games and the Condorcet Jury Theorem." *Games and Economic Behavior*, 25, pp.111-131.

[11] Myerson, Roger (1998b). "Population Uncertainty and Poisson Games." *International Journal of Game Theory*, 27, pp. 375-392.

[12] Myerson, Roger and Weber, Robert (1993). "A Theory of Voting Equilibria." *American Political Science Review*, 87, pp. 102-114.

[13] O'neil, Jeffrey (2007). "Choosing a Runoff Election Threshold." *Public Choice* 131, pp. 351-364.

[14] Perez-Linan, Anibal (2006). "Evaluating Presidential Runoff Elections." *Electoral Studies* 25 (1), pp. 129-146.

[15] Piketty, Thomas (2000). "Voting as Communicating." *Review of Economic Studies*, 67, pp. 169-191.

[16] Riker, William (1982). "The Two-Party System and Duverger's Law: An Essay on the History of Political Science." *American Political Science Review* 84, PP. 1077-1101.

[17] Shugart, Matthew (2006). "Plurality versus Runoff Elections of Presidents: The Mexican Election of 2006 in Comparative Perspective." mimeo Graduate School of International Relations and Pacific Studies, University of California, San Diego.

[18] Shugart, Matthew, and Taagepera, Rein (1994). "Plurality versus Majority Election of Presidents: A Proposal for a "Double Complement Rule"." *Comparative Political Studies* 27, pp. 323-348.

Appendices

Appendix A1: Proofs for Section 3

Property 2 *The magnitudes of the first-round pivot probabilities are:*
***(a) Threshold pivot probability* i/ij,** *i.e.* $x_i^1 = \zeta\left(x_i^1 + x_j^1 + x_k^1\right) > x_j^1 > x_k^1$*:*

$$mag(piv_{i/ij}^1) = \begin{cases} \left(\frac{\tau_j^1+\tau_k^1}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_i^1}{\zeta}\right)^{\zeta} - 1 \ \ if\ \frac{\zeta}{1-\zeta} > \frac{\tau_j^1}{\tau_j^1+\tau_k^1} \geq \frac{1}{2} \\ \left(\frac{\sqrt{\tau_i^1\tau_j^1}}{\zeta}\right)^{2\zeta}\left(\frac{\tau_k^1}{1-2\zeta}\right)^{1-2\zeta} - 1 \ \ if\ \frac{\tau_j^1}{\tau_j^1+\tau_k^1} \geq \frac{\zeta}{1-\zeta} > \frac{1}{2} \\ \left(\frac{2\sqrt{\tau_j^1\tau_k^1}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_i^1}{\zeta}\right)^{\zeta} - 1 \ \ if\ \frac{\zeta}{1-\zeta} > \frac{1}{2} > \frac{\tau_j^1}{\tau_j^1+\tau_k^1} \end{cases} \quad (29)$$

***(b) Above-threshold pivot probability* i/j,** *i.e.* $x_i^1 = x_j^1 \geq \zeta\left(x_i^1 + x_j^1 + x_k^1\right) > x_k^1$*:*

$$mag(piv_{i/j}^1) = \begin{cases} -\left(\sqrt{\tau_i^1} - \sqrt{\tau_j^1}\right)^2 \ \ if\ \sqrt{\tau_i^1\tau_j^1} \geq \tau_k^1\frac{\zeta}{1-2\zeta} \\ \left(\frac{\sqrt{\tau_i^1\tau_j^1}}{\zeta}\right)^{2\zeta}\left(\frac{\tau_k^1}{1-2\zeta}\right)^{1-2\zeta} - 1 \ \ if\ \tau_k^1\frac{\zeta}{1-2\zeta} > \sqrt{\tau_i^1\tau_j^1} \end{cases} \quad (30)$$

***(c) Second-rank pivot probability* ij,** *i.e.* $x_i^1 = x_j^1 < x_k^1 < \zeta\left(x_i^1 + x_j^1 + x_k^1\right)$*:*

$$mag(piv_{ij}^1) = \begin{cases} -\left(\sqrt{\tau_i^1} - \sqrt{\tau_j^1}\right)^2 \ \ if\ 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} > \tau_k^1 > \sqrt{\tau_i^1\tau_j^1} \\ \left(\frac{2\sqrt{\tau_i^1\tau_j^1}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_k^1}{\zeta}\right)^{\zeta} - 1 \ \ if\ \tau_k^1 \geq 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} > \sqrt{\tau_i^1\tau_j^1} \\ 3\left(\tau_i^1\tau_j^1\tau_k^1\right)^{\frac{1}{3}} - 1 \ \ if\ 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} > \sqrt{\tau_i^1\tau_j^1} > \tau_k^1 \end{cases} \quad (31)$$

Proof.

a) Threshold pivot probability i/ij:

A ballot is threshold pivotal i/ij when $x_i^1 = \zeta\left(x_i^1 + x_j^1 + x_k^1\right) > x_j^1 > x_k^1$. Applying Property 1 to compute $mag(piv_{i/ij})$, I have:

$$mag(piv_{i/ij}^1) \quad = \quad \max_x \sum_\psi \frac{x_\psi^1}{n^1}\left(1 - \log(\frac{x_\psi^1}{n^1\tau_\psi^1})\right) - 1 \quad (32)$$

$$s.t. \begin{cases} x_i^1 = \zeta\left(x_i^1 + x_j^1 + x_k^1\right) \\ x_i^1 > x_j^1 \geq x_k^1 \end{cases} \quad (33)$$

If I denote $x_j^1 + x_k^1 = x_{i/ij}^1$, $x_j^1 = \alpha_{i/ij}x_{i/ij}^1$, and $x_k^1 = \left(1 - \alpha_{i/ij}\right)x_{i/ij}^1$, and if I abstract from the second constraint (or if it is not binding) in (33), I find that this is maximized in

$$x^{1*}_{i/ij} = \left(\frac{(1-\zeta)\tau^1_i}{\zeta}\right)^{\zeta}\left(\frac{\tau^1_j}{\alpha}\right)^{\alpha}\left(\frac{\tau^1_k}{1-\alpha}\right)^{1-\alpha}, \tag{34}$$

$$\alpha^*_{i/ij} = \frac{\tau^1_j}{\tau^1_k+\tau^1_j}. \tag{35}$$

Substituting for $x^{1*}_{i/ij}$ and $\alpha_{i/ij}$ in (32) yields what I call the unconstrained magnitude (denoted by the superscript *):

$$mag(piv^{1,*}_{i/ij}) = \left(\frac{\tau^1_j+\tau^1_k}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_i}{\zeta}\right)^{\zeta} - 1. \tag{36}$$

The magnitude of the threshold pivot probability i/ij is equal to the unconstrained one if

$$\frac{\zeta}{1-\zeta} > \frac{\tau^1_j}{\tau^1_j+\tau^1_k} \geq \frac{1}{2}. \tag{37}$$

From (36) and (37), I have that

$$mag(piv^1_{i/ij}) = mag(piv^{1*}_{i/ij}) \text{ if } \frac{\zeta}{1-\zeta} > \frac{\tau^1_j}{\tau^1_j+\tau^1_k} \geq \frac{1}{2}.$$

I still have to compute $mag(piv^1_{i/ij})$ when (37) is not satisfied. Since $\zeta \in \left[\frac{1}{3},1\right)$, I have that $\frac{\zeta}{1-\zeta} \geq \frac{1}{2}$ and then there are two other possible cases:

(i) If $\frac{\tau^1_j}{\tau^1_j+\tau^1_k} > \frac{\zeta}{1-\zeta} \geq \frac{1}{2}$:

In this case, the constraint $x^1_i \geq x^1_j$ is binding. I thus bind the constraint, i.e. $\alpha_{i/ij} = \frac{\zeta}{(1-\zeta)}$, maximize the same problem as in (32). This yields:

$$mag(piv^1_{i/ij}) = \left(\frac{\sqrt{\tau^1_i\tau^1_j}}{\zeta}\right)^{2\zeta}\left(\frac{\tau^1_k}{1-2\zeta}\right)^{1-2\zeta} - 1 \text{ if } \frac{\tau^1_j}{\tau^1_j+\tau^1_k} > \frac{\zeta}{1-\zeta} \geq \frac{1}{2}.$$

(ii) If $\frac{\zeta}{1-\zeta} \geq \frac{1}{2} > \frac{\tau^1_j}{\tau^1_j+\tau^1_k}$:

In this case, the constraint $x^1_j \geq x^1_k$ is binding. We thus bind the constraint, i.e. $\alpha_{i/ij} = 1/2$, and maximize the same problem as in (32). This yields:

$$mag(piv^1_{i/ij}) = \left(\frac{2\sqrt{\tau^1_j\tau^1_k}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_i}{\zeta}\right)^{\zeta} - 1 \text{ if } \frac{\zeta}{1-\zeta} \geq \frac{1}{2} > \frac{\tau^1_j}{\tau^1_j+\tau^1_k}.$$

I have then proven that $mag(piv^1_{i/ij})$ is as defined in (29).

b) Above-threshold pivot probability i/j:

A ballot is above-threshold pivotal i/j when $x^1_i = x^1_j \geq \zeta\left(x^1_i + x^1_j + x^1_k\right) > x^1_k$. Applying Property 1 to compute $mag(piv^1_{i/j})$, I have:

$$mag(piv^1_{i/ij}) = \max_x \sum_\psi \frac{x^1_\psi}{n^1}\left(1 - \log(\frac{x^1_\psi}{n^1\tau^1_\psi})\right) - 1 \tag{38}$$

$$s.t. \begin{cases} x^1_i = x^1_j \\ x^1_i = x^1_j \geq \zeta\left(x^1_i + x^1_j + x^1_k\right) > x^1_k \end{cases} \tag{39}$$

If I denote $x_i^1 = x_j^1 = x_{i/j}^1$, and if I abstract from the second constraint (or if it is not binding) in (39) , I find that this is maximized in

$$\begin{aligned} x_{i/j}^{1*} &= \sqrt{\tau_i^1 \tau_j^1}, \\ x_k^{1*} &= \tau_k^1. \end{aligned}$$

Substituting for x_{ij}^{1*} in (38) yields the unconstrained magnitude (denoted by the superscript *):

$$mag(piv_{i/j}^{1*}) = -\left(\sqrt{\tau_i^1} - \sqrt{\tau_j^1}\right)^2. \tag{40}$$

Since $\zeta \in \left[\frac{1}{3}, 1\right)$, I have that $\frac{\zeta}{1-2\zeta} \geq 1$. Therefore, $\sqrt{\tau_i^1 \tau_j^1} > \tau_k^1 \frac{\zeta}{1-2\zeta}$ implies that $\sqrt{\tau_i^1 \tau_j^1} > \tau_k^1$. This implies that the magnitude of the above-threshold pivot probability i/j is equal to the unconstrained one if

$$\sqrt{\tau_i^1 \tau_j^1} > \tau_k^1 \frac{\zeta}{1-2\zeta}. \tag{41}$$

From (40) and (41), I have that

$$mag(piv_{i/j}^1) = mag(piv_{i/j}^{1*}) \text{ if } \sqrt{\tau_i^1 \tau_j^1} > \tau_k^1 \frac{\zeta}{1-2\zeta}.$$

I still have to compute $mag(piv_{i/j}^1)$ when (41) is not satisfied. There is one other possible case: if $\tau_k^1 \frac{\zeta}{1-2\zeta} > \sqrt{\tau_i^1 \tau_j^1}$. In this case, the constraint $x_i^1 = x_j^1 \geq \zeta\left(x_i^1 + x_j^1 + x_k^1\right)$ is binding. I thus bind the constraint, i.e. $x_{i/j}^1 = x_k^1 \frac{\zeta}{1-2\zeta}$, and maximize the same problem as in (38). This yields:

$$mag(piv_{i/j}^1) = \left(\frac{\sqrt{\tau_i^1 \tau_j^1}}{\zeta}\right)^{2\zeta} \left(\frac{\tau_k^1}{1-2\zeta}\right)^{1-2\zeta} - 1 \text{ if } \tau_k^1 \frac{\zeta}{1-2\zeta} > \sqrt{\tau_i^1 \tau_j^1}.$$

I have then proven that $mag(piv_{i/j}^1)$ is as defined in (30) .

c) Second-rank pivot probabilities ij:

A ballot is second-rank pivotal ij when $x_i^1 = x_j^1 < x_k^1 < \zeta\left(x_i^1 + x_j^1 + x_k^1\right)$. Applying Property 1 to compute $mag(piv_{i/j}^1)$, I have:

$$mag(piv_{i/ij}^1) = \max_x \sum_\psi \frac{x_\psi^1}{n^1}\left(1 - \log\left(\frac{x_\psi^1}{n^1 \tau_\psi^1}\right)\right) - 1 \tag{42}$$

$$s.t. \begin{cases} x_i^1 = x_j^1 \\ x_i^1 = x_j^1 < x_k^1 < \zeta\left(x_i^1 + x_j^1 + x_k^1\right) \end{cases} \tag{43}$$

If I denote $x_i^1 = x_j^1 = x_{ij}^1$, and if I abstract from the second constraint (or if it is not binding) in (43) , I find that this is maximized in

$$\begin{aligned} x_{ij}^{1*} &= \sqrt{\tau_i^1 \tau_j^1}, \\ x_k^{1*} &= \tau_k^1. \end{aligned}$$

Substituting for x_{ij}^{1*} in (42) yields the unconstrained magnitude (denoted by the superscript *):

$$mag(piv_{ij}^{1*}) = -\left(\sqrt{\tau_i^1} - \sqrt{\tau_j^1}\right)^2. \tag{44}$$

The magnitude of the second-rank pivot probability ij is equal to the unconstrained one if

$$2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} > \tau_k^1 > \sqrt{\tau_i^1\tau_j^1} \tag{45}$$

From (44) and (45), I have that

$$mag(piv_{ij}^1) = mag(piv_{ij}^{1*}) \text{ if } 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} > \tau_k^1 > \sqrt{\tau_i^1\tau_j^1}.$$

I still have to compute $mag(piv_{ij}^1)$ when (45) is not satisfied. Since $\zeta \in \left[\frac{1}{3}, 1\right)$, I have that $2\frac{\zeta}{1-\zeta} \geq 1$ and then there are two other possible cases:

(i) If $\tau_k^1 \geq 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} \geq \sqrt{\tau_i^1\tau_j^1}$:

In this case, the constraint $\zeta\left(x_i^1 + x_j^1 + x_k^1\right) \geq x_k^1$ is binding. I thus bind the constraint, i.e. $x_k^1 = \zeta\left(x_i^1 + x_j^1 + x_k^1\right)$, and maximize the same problem as in (42). This yields:

$$mag(piv_{ij}^1) = \left(\frac{2\sqrt{\tau_i^1\tau_j^1}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_k^1}{\zeta}\right)^{\zeta} - 1 \text{ if } \tau_k^1 \geq 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} \geq \sqrt{\tau_i^1\tau_j^1}.$$

(ii) If $2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} \geq \sqrt{\tau_i^1\tau_j^1} > \tau_k^1$:

In this case, the constraint $x_k^1 > x_i^1 = x_j^1$ is binding. We thus bind the constraint, i.e. $x_k^1 = x_i^1 = x_j^1$, and maximize the same problem as in (32). This yields:

$$3\left(\tau_i^1\tau_j^1\tau_k^1\right)^{\frac{1}{3}} - 1 \text{ if } 2\frac{\zeta}{1-\zeta}\sqrt{\tau_i^1\tau_j^1} \geq \sqrt{\tau_i^1\tau_j^1} > \tau_k^1.$$

I have then proven that $mag(piv_{i/ij}^1)$ is as defined in (31). ■

Appendix A2: Proofs for Section 4

Proof of Theorem 1. First, I show that $mag(piv_{AB}^1) \to 0$ when $\tau_A^1 = \tau_B^1 < \tau_C^1 \to \zeta$. Second, I show that $\sigma^1(A|t_A) = \sigma^1(B|t_B) = \sigma^1(C|t_C) = 1$ are equilibrium strategies when $r(t_A) = r(t_B)$, $r(t_C) \to \zeta$ and $\frac{r(t_A)+r(t_B)}{2} < \zeta$. Finally, I show that this equilibrium is generic: it exists for other values of the parameters.

From (14), I have $\tau_C^1\frac{1-\zeta}{\zeta} > 1 - \tau_C^1$. Combining this with (15), I have that

$$\tau_C^1 > 2\frac{\zeta}{1-\zeta}\sqrt{\tau_A^1\tau_B^1} \geq \sqrt{\tau_A^1\tau_B^1}. \tag{46}$$

From Property 2 in Appendix A1 and (46), I can derive $mag(piv_{AB}^1)$:

$$mag(piv_{AB}^1) = \left(\frac{2\sqrt{\tau_A^1\tau_B^1}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_C^1}{\zeta}\right)^{\zeta} - 1. \tag{47}$$

Since $2\sqrt{\tau_A^1 \tau_B^1} \in [0, 1-\tau_C^1]$, $mag(piv_{AB}^1)$ reaches a maximum when $2\sqrt{\tau_A^1 \tau_B^1} = 1 - \tau_C^1$. This means that, for all value of τ_C^1, $mag(piv_{AB}^1)$ is maximized when $\tau_A^1 = \tau_B^1$. When $\tau_A^1 = \tau_B^1$,

$$mag(piv_{AB}) = \left(\frac{1-\tau_C^1}{1-\zeta}\right)^{1-\zeta} \left(\frac{\tau_C^1}{\zeta}\right)^{\zeta} - 1, \tag{48}$$

which is equal to 0 when $\tau_C^1 = \zeta$. Therefore, $mag(piv_{AB}^1) \to 0$ when $\tau_A^1 = \tau_B^1$ and $\tau_C^1 \to \zeta$.

When $\sigma^1(A|t_A) = \sigma^1(B|t_B) = \sigma^1(C|t_C) = 1$, I have $\tau_A^1 = r(t_A)$, $\tau_B^1 = r(t_B)$ and $\tau_C^1 = r(t_C)$. Thus, $mag(piv_{AB}^1) \to 0$ if

$$\begin{aligned} r(t_A) &= r(t_B), \\ r(t_C) &\to \zeta, \text{ and} \\ \frac{r(t_A)+r(t_B)}{2} &< \zeta. \end{aligned} \tag{49}$$

From Property 1, I know that $mag(x^1) \in [-1, 0]$, $\forall x^1$. Therefore, $mag(piv_{AB}^1)$ is the largest magnitude when conditions in (49) are satisfied except if other magnitudes also tend to 0. From Property 2 in Appendix A1, it can be checked easily (but tediously) that this is the case for two other magnitudes: $mag(piv_{C\backslash CA}) \to 0$ and $mag(piv_{C\backslash CB}) \to 0$. Nonetheless, from (9) I have that neither $\Pr\left(piv_{C\backslash CA}\right)$ nor $\Pr\left(piv_{C\backslash CB}\right)$ influence types-t_A and -t_B choice between A and B. Therefore, types-t_A prefer to vote for A and types-t_B prefer to vote for B, i.e. $\sigma(A|t_A) = 1 = \sigma(B|t_B)$ when conditions in (49) are satisfied. Since $mag(piv_{C\backslash CA}) \to 0$ and $mag(piv_{C\backslash CB}) \to 0$, types-$t_C$ prefer to vote for C, i.e. $\sigma(C|t_C) = 1$.

Since all magnitudes are continuous in τ_A^1, τ_B^1 and τ_C^1, there always exist other parameters values such that $\sigma(A|t_A) = \sigma(B|t_B) = \sigma(C|t_C) = 1$ are equilibrium strategies for which C is elected at the first round. The equilibrium is thus generic. ■

Proof of Theorem 2. The proof is in three parts. We prove that Duvergerian equilibria exist (i) when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, (ii) when $\zeta = \frac{1}{2}$, and (iii) when $\zeta \in \left[\frac{1}{2}, 1\right)$. Since proofs of the two Duvergerian equilibria are similar, I only produce the proof for the equilibrium in which all majority types vote for A.

(i) Duvergerian equilibria when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$:

When $\sigma^1(A|t_A) \to 1$ and $\sigma^1(A|t_B) \to 1$, I have $\tau_A^1 \to 1 - r(t_C)$ and $\tau_B^1 \to 0$. This implies that:

$$mag\left(piv_{A/C}^1\right) = \left(\frac{\tau_B^1 + \tau_C^1}{1-\zeta}\right)^{1-\varsigma} \left(\frac{\tau_A^1}{\zeta}\right)^{\zeta} - 1 = \left(\frac{r(t_C)}{1-\zeta}\right)^{1-\zeta} \left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta} - 1 \tag{50}$$

is the only magnitude that can be larger than -1. Indeed, from Property 2 in Appendix A1, I have that:

$$mag\left(piv_{i/j}^1\right) = 3\left(\tau_A^1 \tau_B^1 \tau_C^1\right)^{1/3} - 1 = -1, \forall i, j \neq A, C,$$

and that

$$
\begin{aligned}
mag\left(piv^1_{A/AC}\right) &= \left(\frac{\sqrt{\tau^1_A\tau^1_C}}{\zeta}\right)^{2\zeta}\left(\frac{\tau^1_B}{1-2\zeta}\right)^{1-2\zeta}-1=-1,\\
mag\left(piv^1_{B/BC}\right) &= \left(\frac{2\sqrt{\tau^1_A\tau^1_C}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_B}{\zeta}\right)^{\zeta}-1=-1,\\
mag\left(piv^1_{A/AB}\right) &= \left(\frac{2\sqrt{\tau^1_B\tau^1_C}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_A}{\zeta}\right)^{\zeta}-1=-1,\\
mag\left(piv^1_{B/BA}\right) &= \begin{cases} \text{If } \frac{\zeta}{1-\zeta}\geq\frac{\tau^1_A}{\tau^1_A+\tau^1_C}, & \left(\frac{\tau^1_A+\tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_B}{\zeta}\right)^{\zeta}-1=-1\\ \text{If } \frac{\zeta}{1-\zeta}<\frac{\tau^1_A}{\tau^1_A+\tau^1_C}, & \left(\frac{2\sqrt{\tau^1_B\tau^1_C}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_A}{\zeta}\right)^{\zeta}-1=-1\end{cases},
\end{aligned}
$$

Likewise:

$$
\begin{aligned}
mag\left(piv^1_{AC}\right) &= 3\left(\tau^1_A\tau^1_B\tau^1_C\right)^{1/3}-1=-1,\\
mag(piv^1_{ij}|\omega) &= 2\left(2\sqrt{\tau^1_i\tau^1_j}\tau^1_k\right)^{1/2}-1=-1,\ \forall\{i,j\}\neq\{A,C\},
\end{aligned}
$$

Since $mag(piv^1_{A/C})$ is larger than any other pivot magnitude, for a majority-block voter, $G^1(A|t)$ can be usefully approximated by:

$$
\begin{aligned}
G^1(A|t) &= \Pr\left(piv^1_{A/C}\right)[U(A|t)-U(C|t)]\\
&= \Pr\left(piv^1_{A/C}\right). \qquad (51)
\end{aligned}
$$

I am then in a position to identify a *sufficient* condition for a Duvergerian equilibrium. No voter deviates from $\sigma(A|t_A)=\sigma(A|t_B)=\sigma(C|t_C)=1$ if:

$$G^1(A|t)/G^1(B|t)\underset{n\to\infty}{\to}\infty.$$

Since the magnitude of all the pivot probabilities in favor of B are equal to minus one, this condition is necessarily satisfied when:

$$mag\left(piv^1_{A/C}\right)>-1.$$

From (50), this is true for any $0<r(t_C)<1/2$.

(ii) Duvergerian equilibria when $\zeta=\frac{1}{2}$:

When $\sigma^1(A|t_A)\to 1$ and $\sigma^1(A|t_B)\to 1$, I have $\tau^1_A\to 1-r(t_C)$ and $\tau^1_B\to 0$. This implies that:

$$mag\left(piv^1_{A/AC}\right)=-\left(\sqrt{\tau^1_A}-\sqrt{\tau^1_C}\right)^2=-\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^2$$

is the only magnitude that can be larger than -1. Indeed, from Property 2 in Appendix A1 , I have that:

$$mag\left(piv^1_{i/ij}\right)=2\left(2\sqrt{\tau^1_j\tau^1_k}\tau^1_i\right)^{1/2}-1=-1,$$

for any set of candidates $\{i,j,k\} \neq \{A,C,B\}$. Likewise:

$$\begin{aligned} mag\left(piv^1_{AC}\right) &= 3\left(\tau^1_A\tau^1_B\tau^1_C\right)^{1/3} - 1 = -1 \text{ and} \\ mag\left(piv^1_{ij}\right) &= 2\left(2\sqrt{\tau^1_i\tau^1_j}\tau^1_k\right)^{1/2} - 1 = -1, \forall\{i,j\} \neq \{A,C\}. \end{aligned}$$

Note that a ballot cannot be above-threshold pivotal when $\zeta \geq \frac{1}{2}$.

Since $mag(piv^1_{A/AC})$ is larger than any other pivot magnitude, for a majority-block voter, $G^1(A|t)$ can be usefully approximated by:

$$\begin{aligned} G^1(A|t) &= \Pr\left(piv^1_{A/AC}\right)[U(A|t) - U(A,C|t)] \\ &= \Pr(C \mid \{A,C\})\Pr\left(piv^1_{A/AC}\right). \end{aligned} \tag{52}$$

Remark the first element in (52): if the second round opposes A to C, and A wins that round, then being first-round pivotal has no value. This is why being first-round pivotal is only valuable with probability $\Pr(C \mid \{A,C\})$.

Next, remark that the distribution of A and C votes are identical in the first and second rounds. Hence:

$$\Pr(C \mid \{A,C\}) \geq \Pr\left(piv^1_{A/AC}\right)$$

and then

$$G^1(A|t) \geq \Pr\left(piv^1_{A/AC}\right)^2,$$

which provides a lower bound to $G^1(A|t)$.

I am finally in a position to identify a *sufficient* condition for a Duvergerian equilibrium. No voter deviates from $\sigma^1(A|t_A) = \sigma^1(A|t_B) = \sigma^1(C|t_C) = 1$ if:

$$G^1(A|t)/G^1(B|t) \underset{n\to\infty}{\rightarrow} \infty.$$

Since the magnitude of all the pivot probabilities in favor of B are equal to minus one, this condition is necessarily satisfied when:

$$2mag\left(piv^1_{A/AC}\right) > -1.$$

From $mag\left(piv^1_{A/AC}\right) = -\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2$, this condition boils down to:

$$\sqrt{1-r(t_C)} - \sqrt{r(t_C)} < \sqrt{1/2},$$

or: $r(t_C) > 0.06699$.

(iii) Duvergerian equilibria when $\zeta \in \left[\frac{1}{2}, 1\right)$:

When $\sigma^1(A|t_A) \to 1$ and $\sigma^1(A|t_B) \to 1$, I have $\tau^1_A \to 1 - r(t_C)$ and $\tau^1_B \to 0$. This implies that:

$$mag\left(piv^1_{A/AC}\right) = \left(\frac{\tau^1_B + \tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_A}{\zeta}\right)^{\zeta} - 1 = \left(\frac{r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta} - 1 \tag{53}$$

is the only magnitude that can be larger than -1. Indeed, from Property 2 in Appendix A1, I have that:

$$\begin{aligned} mag\left(piv^1_{B/BA}\right) &= \left(\frac{\tau^1_A+\tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_B}{\zeta}\right)^{\zeta}-1=-1, \\ mag\left(piv^1_{i/ij}\right) &= \left(\frac{2\sqrt{\tau^1_k\tau^1_j}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_i}{\zeta}\right)^{\zeta}-1=-1,\ \forall\{i,j\}\neq\{B,A\},\{A,C\} \\ mag\left(piv^1_{AC}\right) &= 3\left(\tau^1_A\tau^1_B\tau^1_C\right)^{1/3}-1=-1, \\ mag\left(piv^1_{ij}\right) &= \left(\frac{2\sqrt{\tau^1_i\tau^1_j}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_k}{\zeta}\right)-1=-1,\ \forall i,j\neq A,C. \end{aligned}$$

Remind that a ballot cannot be above the threshold pivotal when $\zeta\geq\frac{1}{2}$.

Since $mag(piv^1_{A/AC})$ is larger than any other pivot magnitude, for a majority-block voter, $G^1(A|t)$ can be usefully approximated by:

$$\begin{aligned} G^1(A|t) &= \Pr\left(piv^1_{A/AC}\right)[U(A|t)-U(A,C|t)] \\ &= \Pr(C\mid\{A,C\})\Pr\left(piv^1_{A/AC}\right). \end{aligned} \tag{54}$$

I am finally in a position to identify a *sufficient* condition for a Duvergerian equilibrium. No voter deviates from $\sigma^1(A|t_A)=\sigma^1(A|t_B)=\sigma^1(C|t_C)=1$ if:

$$G^1(A|t)/G^1(B|t)\underset{n\to\infty}{\to}\infty.$$

Since the magnitude of all the pivot probabilities in favor of B are equal to minus one, this condition is necessarily satisfied when:

$$mag\left(\Pr(C\mid\{A,C\})\right)+mag\left(piv^1_{A/AC}\right)>-1.$$

From (53) and knowing that $mag\left(\Pr(C\mid\{A,C\})\right)=2\sqrt{(1-r(t_C))\,r(t_C)}-1$, this condition boils down to:

$$2\sqrt{(1-r(t_C))\,r(t_C)}+\left(\frac{r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta}\geq 1. \tag{55}$$

Knowing that

$$\begin{aligned} \frac{\partial\left(\left(\frac{1-r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t_C)}{\zeta}\right)^{\zeta}\right)}{\partial\zeta} &= \left(\frac{r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta}\left(1-\log\left(\frac{r(t_C)}{1-\zeta}\right)\right)+ \\ &\quad \left(\frac{r(t_C)}{1-\zeta}\right)^{1-\varsigma}\left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta}\left(\log\left(\frac{1-r(t_C)}{\zeta}\right)-1\right), \end{aligned}$$

I have that

$$\begin{aligned} \frac{\partial\left(\left(\frac{1-r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t_C)}{\zeta}\right)^{\zeta}\right)}{\partial\zeta} &> 0 \text{ if } \zeta<1-r(t_C), \\ &= 0 \text{ if } \zeta=1-r(t_C), \\ &< 0 \text{ if } \zeta>1-r(t_C), \end{aligned} \tag{56}$$

and then that

$$\min_{\zeta}\left(\frac{r(t_C)}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t_C)}{\zeta}\right)^{\zeta} > \min\left\{\underbrace{2\sqrt{(1-r(t_C))\,r(t_C)}}_{\text{for }\zeta=1/2}, \underbrace{1-r(t_C)}_{\text{for }\zeta=1}\right\}. \tag{57}$$

There are then two cases to consider: (i) $2\sqrt{(\ 1-r(t_C))\ \ r(t_C)} < 1-r(t_C)$ and (ii) $2\sqrt{(1-r(t_C))\,r(t_C)} \geq 1-r(t_C)$. In case (i) I have that $\min\left\{2\sqrt{(1-r(t_C))\,r(t_C)}, 1-r(t_C)\right\} = 2\sqrt{(1-r(t_C))\,r(t_C)}$ and then that

$$\left(\frac{1-r(t_C)}{1-\zeta}\right)^{1-\varsigma}\left(\frac{r(t_C)}{\zeta}\right)^{\zeta} > 2\sqrt{(1-r(t_C))\,r(t_C)}. \tag{58}$$

Knowing that if $r(t_C) > 0.06699$, then $2\sqrt{(1-r(t_C))\,r(t_C)} > \frac{1}{2}$, I have from (58) that (55) is satisfied if $r(t_C) > Z$ with $Z < 0.06699$. In case (ii) I have that $\min\left\{2\sqrt{(1-r(t_C))\,r(t_C)}, 1-r(t_C)\right\} = 1-r(t_C)$. Since $(1-r(t_C)) > \frac{1}{2}$, both $\left(\frac{1-r(t_C)}{1-\zeta}\right)^{1-\varsigma}\left(\frac{r(t_C)}{\zeta}\right)^{\zeta}$ and $2\sqrt{(1-r(t_C))\,r(t_C)}$ are larger than $\frac{1}{2}$. Therefore, (55) is always strictly satisfied. ■