Connections among farsighted agents †

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Abstract

We adopt the notion of pairwise farsightedly stable set to predict which networks may be formed among farsighted players. We investigate in the symmetric connections model and the buyer-seller networks whether the pairwise farsightedly stable sets of networks coincide with the set of pairwise stable networks and the set of strongly efficient networks. We also provide some primitive conditions on value functions and allocation rules so that the set of strongly efficient networks is the unique pairwise farsightedly stable set.

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1 Introduction

The organization of individual agents into networks and groups or coalitions plays an important role in the determination of the outcome of many social and economic interactions. For instance, networks of personal contacts are important in obtaining information on goods and services, like product information or information about job opportunities. Many commodities are traded through networks of buyers and sellers. A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that individuals do not benefit from altering the structure of the network. An example of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996).[1] Their approach is static and myopic. Individuals are not forward-looking in the sense that they do not forecast how others might react to their actions. For instance, individuals might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original individuals.

Herings, Mauleon and Vannetelbosch (2009) have proposed the notion of pairwise farsightedly stable sets of networks that predicts which networks one might expect to emerge in the long run when players are farsighted.[2] A set of networks G is pairwise farsightedly stable (i) if all possible farsighted pairwise deviations from any network $g \in G$ to a network outside G are deterred by the threat of ending worse off or equally well off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set,[3] and (iii) if there is no proper subset of G satisfying Conditions (i) and (ii). A non-empty pairwise farsightedly stable set always exists. Herings, Mauleon and Vannetelbosch (2009) have provided a full characterization of unique pairwise farsightedly stable sets of networks. Contrary to other pairwise concepts, pairwise farsighted stability yields a Pareto dominant network, if it exists, as the unique outcome. Finally, we study the relationship between pairwise farsighted stability and other concepts such as the largest pairwise consistent set and the von Neumann-Morgenstern pairwise farsightedly stable

[1] Jackson (2003, 2005) provides surveys of models of network formation.

[2] Other approaches to farsightedness in network formation are suggested by the work of Xue (1998), Herings, Mauleon, and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Page, Wooders and Kamat (2005), Dutta, Ghosal, and Ray (2005), and Page and Wooders (2009).

[3] A farsighted improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network.

set.[4]

In this paper we adopt the notion of pairwise farsightedly stable set to predict which networks may be formed among farsighted players in Jackson and Wolinsky (1996) symmetric connections model, in Corominas-Bosch (2004) model of trading networks with bilateral bargaining, and in Kranton and Minehart (2001) model of buyer-seller networks. We investigate whether the pairwise farsightedly stable sets of networks coincide with the set of pairwise stable networks and the set of strongly efficient networks. Finally, we also provide some primitive conditions on value functions and allocation rules so that the set of strongly efficient networks is the unique pairwise farsightedly stable set.

The paper is organized as follows. In Section 2 we introduce some notations and basic properties and definitions for networks. In Section 3 we define the notion of pairwise farsightedly stable set of networks. In Section 4 we consider the symmetric connections model. In Section 5 we consider the buyer-seller networks. We look at the relationship between farsighted stability and efficiency of networks in Section 6.

2 Networks

Let $N = \{1, \ldots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. We write $ij \in g$ to indicate that i and j are linked under the network g. Let g^N be the collection of all subsets of N with cardinality 2, so g^N is the complete network. The set of all possible networks or graphs on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \mid \exists j$ such that $ij \in g\}$ be the set of players who have at least one link in the network g. A path in a network $g \in \mathbb{G}$ between i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = 1$ and $i_K = j$. A non-empty network $h \subseteq g$ is a component of g, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$. Knowing the components of a network, we can partition the

[4] Notice that any von Neumann-Morgenstern pairwise farsightedly stable set is a pairwise farsightedly stable set. But, von Neumann-Morgenstern pairwise farsightedly stable set may fail to exist. Pairwise farsightedly stable sets have no relationship to either largest pairwise consistent sets or sets of pairwise stable networks.

players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by the network g.[5]

A value function is a function $v : \mathbb{G} \to \mathbb{R}$ that keeps track of how the total societal value varies across different networks. The set of all possible value functions is denoted by $\mathcal{V}$. An allocation rule is a function $Y : \mathbb{G} \times \mathcal{V} \to \mathbb{R}^N$ that keeps track of how the value is allocated or distributed among the players forming a network. It satisfies $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g.

Jackson and Wolinsky (1996) have proposed a number of basic properties of value and allocation functions. A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in \mathbb{G}$. Component additive value functions are the ones for which the value of a network is the sum of the value of its components. An allocation rule Y is *component balanced* if for any component additive $v \in \mathcal{V}$, $g \in \mathbb{G}$, and $h \in C(g)$, we have $\sum_{i \in N(h)} Y_i(h, v) = v(h)$. Component balancedness only puts conditions on Y for v's that are component additive, so Y can be arbitrary otherwise. Given a permutation of players π and any $g \in \mathbb{G}$, let $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\}$. Thus, g^π is a network that is identical to g up to a permutation of the players. A value function is *anonymous* if for any permutation π and any $g \in \mathbb{G}$, $v(g^\pi) = v(g)$. Given a permutation π, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in \mathbb{G}$. An allocation rule Y is *anonymous* if for any $v \in \mathcal{V}$, $g \in \mathbb{G}$, and permutation π, we have $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.[6]

An allocation rule that is component balanced and anonymous is the *componentwise egalitarian allocation rule*. For a component additive v and network g, the componentwise egalitarian allocation rule Y^{ce} is such that for any $h \in C(g)$ and each $i \in N(h)$, $Y_i^{ce}(g, v) = v(h)/\#N(h)$. For a v that is not component additive, $Y^{ce}(g, v) = v(g)/n$ for all g; thus, Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive.

In evaluating societal welfare, we may take various perspectives. A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in \mathbb{G}$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with at least one strict inequality. A network $g \in \mathbb{G}$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in \mathbb{G}$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

The network-theoretic literature uses two different notions of a deviation by a coalition. *Pairwise deviations* (Jackson and Wolinsky, 1996) are deviations involving a single link

[5] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

[6] Anonymous value functions are those such that the architecture of a network matters, but not the labels of individuals. Anonymity of an allocation rule requires that if only the labels of the agents change and the value generated by networks changes in an exactly corresponding fashion, then the allocation only changes according to the relabeling.

at a time. Moreover, link addition is bilateral (two players that would be involved in the link must agree to adding the link), link deletion is unilateral (at least one player involved in the link must agree to deleting the link), and network changes take place one link at a time. *Coalitionwise deviations* (Jackson and van den Nouweland, 2005) are deviations involving several links and some group of players at a time. Link addition is bilateral, link deletion is unilateral, and multiple link changes can take place at a time. Whether a pairwise deviation or a coalitionwise deviation makes more sense will depend on the setting within which network formation takes place.

We will restrict our analysis to pairwise deviations. A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that agents do not benefit from altering the structure of the network. A weak version of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Formally, a network g is pairwise stable with respect to value function v and allocation rule Y if

(i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and

(ii) for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.

We say that g' is adjacent to g if $g' = g + ij$ or $g' = g - ij$ for some ij. A network g' defeats g if either $g' = g - ij$ and $Y_i(g', v) > Y_i(g, v)$ or $Y_j(g', v) > Y_j(g, v)$, or if $g' = g + ij$ with $Y_i(g', v) \geq Y_i(g, v)$ and $Y_j(g', v) \geq Y_j(g, v)$ with at least one inequality holding strictly. Pairwise stability is equivalent to the statement of not being defeated by another network.[7]

3 Pairwise farsightedly stable sets of networks

A *farsighted improving path* is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is

[7] Jackson and van den Nouweland (2005) have proposed a refinement of pairwise stability where coalitionwise deviations are allowed: the strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component.

deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network. We now introduce the formal definition of a farsighted improving path.

Definition 1 *A farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of graphs $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either:*

(i) $g_{k+1} = g_k - ij$ *for some ij such that* $Y_i(g_K, v) > Y_i(g_k, v)$ *or* $Y_j(g_K, v) > Y_j(g_k, v)$*, or*

(ii) $g_{k+1} = g_k + ij$ *for some ij such that* $Y_i(g_K, v) > Y_i(g_k, v)$ *and* $Y_j(g_K, v) \geq Y_j(g_k, v)$*.*

If there exists a farsighted improving path from g to g', then we write $g \to g'$. For a given network g, let $F(g) = \{g' \in G \mid g \to g'\}$. This is the set of networks that can be reached by a farsighted improving path from g. Thus, $g \to g'$ means that g' is the endpoint of at least one farsighted improving path from g. Notice that $F(g)$ may contain many networks and that a network $g' \in F(g)$ might be the endpoint of several farsighted improving paths starting in g. Since we are interested in stability of networks, there will be no need to specify on which particular path players eventually agree. Rather $F(g)$ represents the networks that could possibly be reached by farsighted players when starting in g, and our concept of stability takes these possible end networks into account in a way that we will make precise in Definition 2.

We now introduce a solution concept due to Herings, Mauleon and Vannetelbosch (2009), the pairwise farsightedly stable set. The definition corresponds to the one of a pairwise myopically stable set with myopic deviations replaced by farsighted deviations. It is obtained by requiring the deterrence of farsighted external deviations, farsighted external stability, and minimality. More precisely, a set of networks G is pairwise farsightedly stable if (i) all possible *pairwise deviations* from any network $g \in G$ to a network outside G are deterred by a credible threat of ending worse off or equally well off, (ii) there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) there is no proper subset of G satisfying Conditions (i) and (ii). Formally, pairwise farsightedly stable sets are defined as follows.

Definition 2 *A set of networks $G \subseteq \mathbb{G}$ is pairwise farsightedly stable with respect v and Y if*

(i) $\forall\ g \in G$,

(ia) $\forall\ ij \notin g$ *such that* $g+ij \notin G$, $\exists\, g' \in F(g+ij) \cap G$ *such that* $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ *or* $Y_i(g', v) < Y_i(g, v)$ *or* $Y_j(g', v) < Y_j(g, v)$,

(ib) $\forall\, ij \in g$ *such that* $g - ij \notin G$, $\exists\, g', g'' \in F(g-ij) \cap G$ *such that* $Y_i(g', v) \leq Y_i(g, v)$ *and* $Y_j(g'', v) \leq Y_j(g, v)$,

(ii) $\forall g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$.

(iii) $\nexists\, G' \subsetneq G$ *such that* G' *satisfies Conditions (ia), (ib), and (ii).*

Condition (ia) in Definition 2 captures that adding a link ij to a network $g \in G$ that leads to a network outside of G, is deterred by the threat of ending in g'. Here g' is such that there is a farsighted improving path from $g + ij$ to g'. Moreover, g' belongs to G, which makes g' a credible threat. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) in Definition 2 requires external stability and implies that the networks within the set are robust to perturbations. From any network outside G there is a farsightedly stable path leading to some network in G.[8] Condition (ii) implies that if a set of networks is pairwise farsightedly stable, it is non-empty. Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 2. This motivates the requirement of a minimality condition, namely Condition (iii). Herings, Mauleon and Vannetelbosch (2009) have shown that a pairwise farsightedly stable set of networks always exists.

A network g strictly Pareto dominates all other networks if g is such that for all $g' \in \mathbb{G} \setminus \{g\}$ it holds that, for all i, $Y_i(g, v) > Y_i(g', v)$. Although the network that strictly Pareto dominates all others is pairwise stable, there might be many more pairwise stable networks. Herings, Mauleon and Vannetelbosch (2009) have shown that, if there is a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique pairwise farsightedly stable set. Thus, pairwise farsighted stability singles out the Pareto dominating network as the unique pairwise farsightedly stable set.

4 Symmetric connections models

Players form links with each other in order to exchange information. If player i is connected to player j by a path of t links, then player i receives a payoff of δ^t from his indirect connection with player j. It is assumed that $0 < \delta < 1$, and so the payoff δ^t decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost c to both i and j. This cost can be interpreted as the

[8] There are some random dynamic models of network formation that are based on incentives to form links such as Watts (2001), Jackson and Watts (2002), and Tercieux and Vannetelbosch (2006). These models aim to use the random process to select from the set of pairwise stable networks.

time a player must spend with another player in order to maintain a direct link. Player i's payoff from a network g is given by

$$Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j:ij \in g} c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j).

Jackson and Wolinsky (1996) proved the following two results.

Proposition 1 (Jackson and Wolinsky, 1996) *The unique strongly efficient network in the symmetric connections model is*

(i) the complete network g^N if $c < \delta(1-\delta)$,

(ii) a star encompassing everyone if $\delta(1-\delta) < c < \delta + ((N-2)/2)\delta^2$, and

(iii) the empty network g^e if $\delta + ((N-2)/2)\delta^2 < c$.

Proposition 2 (Jackson and Wolinsky, 1996) *In the symmetric connections model with $Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j:ij \in g} c$:*

(i) For $c < \delta(1-\delta)$, the unique pairwise stable network is the complete network g^N.

(ii) For $\delta(1-\delta) < c < \delta$, a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable network.

(iii) For $\delta < c$, any pairwise stable network which is non-empty is such that each player has at least two links and thus is inefficient.

These two results show that there is a conflict between efficiency and pairwise stability for a large range of the parameters. Indeed, only for $c < \delta(1-\delta)$, there is no conflict between the efficient and the pairwise stable networks. When $\delta(1-\delta) < c < \delta$, the efficient network is pairwise stable, but there are other pairwise stable networks that are not efficient. For $\delta < c < \delta + ((N-2)/2)\delta^2$, the efficient network is never pairwise stable. And, finally, for $\delta + ((N-2)/2)\delta^2 < c$, the efficient network is pairwise stable, but there could be other pairwise stable networks that are not efficient.

In Figure 1 we have depicted the 4$-$player case where (i) for $c < \delta(1-\delta)$, the complete network is the unique pairwise stable network, (ii) for $\delta(1-\delta) < c < \delta$, the star networks are pairwise stable but they are not the only pairwise stable networks (if $\delta - \delta^3 < c < \delta$, then a line is also stable, and if $c < \delta - \delta^3$, then a circle is also stable), and (iii) for $c > \delta$, the empty network is the unique pairwise stable network.

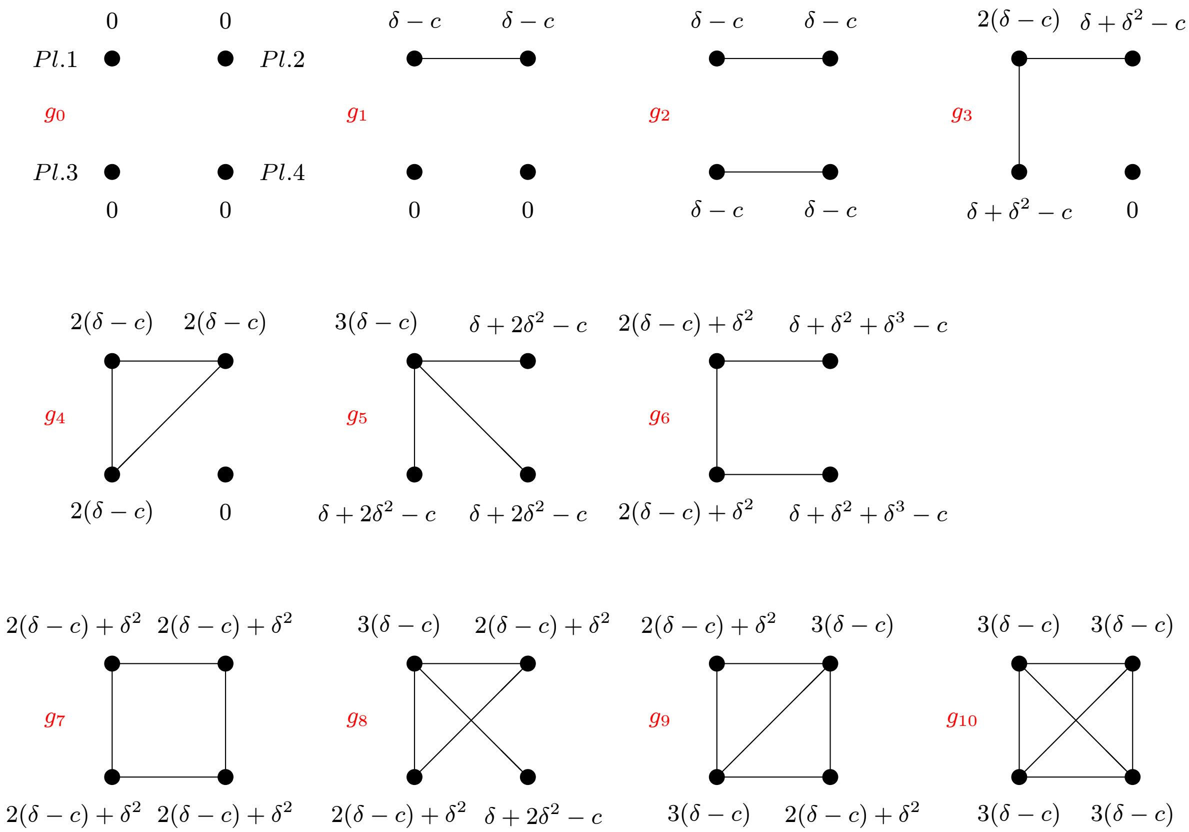

Figure 1 : The symmetric connections model with four players.

Applying the concept of pairwise farsightedly stable sets to the symmetric connections model with four players, we obtain the following. First we consider the case $c < \delta(1-\delta)$. It holds that the complete network $g_{10} \in F(g)$ for any other network $g \neq g_{10}$, and $F(g_{10}) = \emptyset$. Now it follows by Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) that $\{g_{10}\}$ is the unique pairwise farsightedly stable set.

Next we consider the case $\delta(1-\delta) < c < \delta$. It holds that $F(g_0) = \{g_1, g_2, g_3, g_5, g_6$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_1) = \{g_2, g_3, g_5, g_6$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_2) = \{g_3, g_5, g_6$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_3) = \{g_5, g_6$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_4) = \{g_3, g_5, g_6, g_8$ and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_5) = \{$ the other star networks, g_6 with the hub player not at the extreme of the line, g_6 with the hub player at the extreme of the line if $(\delta - c) - \delta^3 < \delta^2 - (\delta - c)$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_6) = \{g_5$, the other lines, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_7) = \{g_5$, and g_6 if $\delta(1-\delta^2) < c\}$, $F(g_8) = \{g_5, g_6$, and g_7 if $\delta(1-\delta^2) > c\}$, $F(g_9) = \{g_5, g_6, g_8$, and g_7 if $\delta(1-\delta^2) > c\}$ and $F(g_{10}) = \{g_5, g_6, g_8, g_9$, and g_7 if $\delta(1-\delta^2) > c\}$. By a repeated application of Theorem 4 in Herings, Mauleon and Vannetelbosch (2009), it follows that each star $\{g_5\}$ is a pairwise farsightedly stable set. But they are not the unique pairwise farsightedly stable sets of networks. Indeed, if $\delta(1-\delta^2) < c$, every line $\{g_6\}$ is a pairwise farsightedly stable set; but if $\delta(1-\delta^2) > c$, the circle $\{g_7\}$ is a pairwise farsightedly stable set. Moreover, in this last case with $\delta(1-\delta^2) > c$, we also have that every line $\{g_6\}$ is a pairwise farsightedly stable set if $(\delta-c)-\delta^3 < \delta^2-(\delta-c)$, and every two lines with the role of the players reversed is a pairwise farsightedly stable set only if $(\delta-c)-\delta^3 > \delta^2-(\delta-c)$.

Finally, we examine the case $c > \delta$. One may verify that $F(g_0) = \{g_6 \text{ if } \delta + \frac{\delta^2}{2} > c\}$, $F(g_1) = \{g_0, g_6 \text{ if } \delta + \frac{\delta^2}{2} > c\}$, $F(g_2) = \{g_0, g_1, g_6 \text{ if } \delta + \delta^2 > c\}$, $F(g_3) = \{g_0, g_1, g_6 \text{ if } \delta + \frac{\delta^2}{2} > c\}$, $F(g_4) = \{g_0, g_1, g_3, g_6 \text{ if } \delta + \frac{\delta^2}{2} > c\}$, $F(g_5) = \{g_0, g_1, g_3, g_6 \text{ if } \delta + \frac{\delta^2}{2} > c\}$, $F(g_6) = \{g_3$, the other lines, and g_0 and g_1 if $\delta + \frac{\delta^2}{2} < c\}$, $F(g_7) = \{g_3, g_6$, and g_0 and g_1 if $\delta + \frac{\delta^2}{2} < c\}$, $F(g_8) = \{g_0, g_1, g_3, g_4, g_5, g_6\}$, $F(g_9) = \{g_0, g_1, g_3, g_4, g_5, g_6, g_8\}$ and $F(g_{10}) = \{g_0, g_1, g_3, g_4, g_5, g_6, g_8, g_9\}$. It follows by Theorem 4 in Herings, Mauleon and Vannetelbosch (2009) that $\{g_0\}$ is the unique pairwise farsightedly stable set if $\delta + \frac{\delta^2}{2} < c$; but any set $\{g_6\}$ containing a given line and all lines that can be obtained by means of links' permutations that make players remaining with the same degree of centrality as in the given line, is a pairwise farsightedly stable set if $\delta + \frac{\delta^2}{2} > c > \delta$. In this last case, when $\delta + \frac{\delta^2}{2} > c > \delta$, also the sets containing g_0 and some of the networks with a form like g_3 are pairwise farsightedly stable sets of networks.

This example shows that when $c < \delta(1-\delta)$, the unique efficient and pairwise stable network (the complete one) is the unique pairwise farsightedly stable set of networks and, then, there is no conflict between efficiency and pairwise stability or between efficiency and pairwise farsighted stability. When $\delta(1-\delta) < c < \delta$, a star encompassing all players is pairwise stable and a pairwise farsightedly stable set of networks, but not necessarily the unique pairwise stable network or the unique pairwise farsightedly stable set of networks. Moreover, when the star network is pairwise stable or pairwise farsightedly stable it is also efficient, but there are still values of the parameters (see Propositions 1 and 2 in Jackson and Wolinsky, 1996) for which the star network is efficient but neither pairwise stable or pairwise farsightedly stable. When $c > \delta$, the empty network is always pairwise stable and it always belongs to some pairwise farsightedly stable sets of networks. Indeed, when $c > \delta + \frac{\delta^2}{2}$, the empty network is also the unique pairwise farsightedly stable set, and when $\delta < c < \delta + \frac{\delta^2}{2}$, the sets containing the empty network and some of the networks with a form like g_3 are pairwise farsightedly stable sets of networks. Also, every set of lines such that each player has the same degree of centrality on all the lines in the set are pairwise farsightedly stable sets of networks. Thus, when the empty network is efficient (for $c > \delta + \delta^2$) is also pairwise stable and pairwise farsightedly stable, but there are still parameters values for which the empty network is pairwise stable and pairwise farsightedly stable but not efficient. Therefore, the introduction of pairwise farsighted stability do not eliminate the conflict between stability and efficiency in the symmetric connections model with four players.

Next proposition shows that these results also hold in the symmetric connections model with N players.

Proposition 3 *In the symmetric connections model with* $Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j:ij \in g} c$,

(i) For $c < \delta(1-\delta)$*, the unique pairwise farsightedly stable set of networks is the complete network* $\{g^N\}$.

(ii) For $\delta(1-\delta) < c < \delta$*, every star network encompassing all players is a pairwise farsightedly stable set of networks, but they are not necessarily the unique pairwise farsightedly stable sets of networks.*

(iii) For $c > \delta$*, the empty network is the unique pairwise farsightedly stable set of networks if* $c > \delta + \frac{(N-2)\delta^2}{2}$*. Otherwise, if* $\delta < c < \delta + \frac{(N-2)\delta^2}{2}$*, the empty network is not necessarily the unique pairwise farsightedly stable set.*

Proof.

(i) Assume $c < \delta(1-\delta)$. Since $\delta < 1$, we have that $(\delta - c) > \delta^2 > \delta^3 > ... > \delta^{n-1}$. Thus, any two agents who are not directly connected benefit from forming a link. In this case, the complete network g^N strictly Pareto dominates all other networks. That is, for every $g \in \mathbb{G} \setminus g^N$ we have that, for all i, $Y_i(g^N) > Y_i(g)$. So, applying Theorem 7 in Herings, Mauleon and Vannetelbosch (2009), we have that $\{g^N\}$ is the unique pairwise farsightedly stable set.

(ii) Assume $\delta(1-\delta) < c < \delta$. Since $\delta^2 > (\delta - c)$, and $\delta^2 > \delta^3 > ... > \delta^{n-1}$, each agent prefers an indirect link at a distance of two to any direct link and to any indirect link at a distance greater than two. In a star network encompassing all players g^s there is $n-1$ links connecting one given player i to any other player $j \in N$, $j \neq i$. Then, the payoff of the hub player i is $Y_i(g^s) = (n-1)(\delta - c)$ and the payoff of any spoke player j, $j \neq i$, is $Y_j(g^s) = (\delta - c) + (n-2)\delta^2$. Notice that the payoff of the spoke players is the maximum payoff a player can get in any network $g \in \mathbb{G}$. In order to prove that every star network encompassing all players $\{g^s\}$ is a pairwise farsightedly stable set, we will prove that $g^s \in F(g)$ for any $g \neq g^s$ and, hence, Theorem 4 in Herings et al. (2008) applies.

- Consider first any network g containing at most $n-1$ links. Starting from the empty network g^e, it is straightforward to construct a farsightedly improving path leading to g^s in order to have that $g^s \in F(g^e)$. Take the hub player i and any other player and form the link between them. Then, add successively the links between the hub player and any other player until g^s is formed. Starting from any network g^k with $k \leq n-1$ links that is not a star network, notice that the payoff of any player $j \neq i$ is smaller than the payoff she derives at g^s. Then, let to every player $j \neq i$ to delete successively all its links until the empty network is reached. From g^e, add successively the links between player i and the rest of the players until g^s is formed. Obviously, $g^s \in F(g^k)$. Consider now any other star network $g^{s'}$ different than g^s with $j \neq i$ as hub player. From $g^{s'}$, let the hub player j to delete successively any link between her and any other player until the empty network is reached. From g^e, add successively the links between player i and the rest of the players until g^s is formed. Obviously, $g^s \in F(g^{s'})$ because player j prefers g^s to $g^{s\prime}$ and the remaining players (including player i) also prefer g^s to the network they were facing before forming the necessary links to form g^s.

- Consider next any network g containing more than $n-1$ links. In such network g, there is always at least a player $j \neq i$ with more than one direct links and that would like to delete all its links looking forward to g^s. From g, let such players to delete successively each of its links. Notice that the resulting network g' should have less than $n-1$ links. From g', delete successively all the links until the empty network is reached. From g^e, add successively the links between player i and the rest of the players until g^s is formed. Thus, $g^s \in F(g)$ and Theorem 4 in Herings, Mauleon and Vannetelbosch (2009) applies.

(iii.1) Assume first that $c > \delta + \frac{(N-2)\delta^2}{2}$. In order to show that the empty network (with a payoff of 0 for all players) is the unique pairwise farsightedly stable set of networks, we need to show that Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) applies; i.e., that $g^e \in F(g)$ for all $g \neq g^e$ and that $F(g^e) = \varnothing$.

Since $c > \delta + \frac{(N-2)\delta^2}{2}$, the empty network is the unique efficient network. This implies that in any other network g, there is some player with a negative payoff that prefers the empty network and hence, we have that $g \notin F(g^e)$. Let the players with a negative payoff at g to delete successively all its links. Since in any resulting network g', there is some player prefering the empty network, letting such players deleting successively all its links we finally arrive to the empty network g^e, and $g^e \in F(g)$. Thus, $g^e \in F(g)$ for all $g \neq g^e$ and Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) applies.

(iii.2) Assume now that $\delta < c < \delta + \frac{(N-2)\delta^2}{2}$. In this case, the empty network is not more the efficient network (a star encompassing everyone is the efficient network). However, there are still some parameter values for which the empty network is a pairwise farsightedly stable set. Indeed, the necessary and sufficient condition in order to have that $g^e \in F(g)$ for all $g \neq g^e$ is that, in each other network g, there should be a player with a negative payoff that would like to delete her links in order to go to g^e (and then notice that each other network g is such that $g \notin F(g^e)$). That is, if the $\min_i Y_i(g) < 0$ for some i for all $g \neq g^e$, then such player(s) would delete all her links looking forward to g^e from any other network g. Hence, Corollary 1 in Herings, Mauleon and Vannetelbosch (2009) applies, and $\{g^e\}$ is the unique pairwise farsightedly stable set. Notice that the value of c for which the above condition holds is strictly smaller than $\delta + \frac{(N-2)\delta^2}{2}$. On the contrary, if $\min_i Y_i(g) > 0$ for all i for some $g \neq g^e$, we have that $g^e \notin F(g)$ and then $\{g^e\}$ is not a pairwise farsightedly stable set. However, one could still have that a set of networks containing among others the empty network is a pairwise farsightedly stable set of networks (see the above example with 4 players). ■

Proposition 3 shows that replacing myopic by farsighted players in the symmetric connections model does not eliminate the conflict between efficiency and pairwise (farsighted) stability but, sometimes, it could help to reduce the conflict (for example, for $\delta + ((N-2)/2)\delta^2 < c$, the unique efficient network is also the unique pairwise farsightedly stable set of networks). Regarding the relationship between pairwise stability and pairwise farsighted stability, one can see that the concept of pairwise stability is quite robust to the introduction of farsighted players because, for a large range of parameters, one can show that some of the pairwise stable networks are also a pairwise farsightedly stable set of networks. However, there are still some parameters values for which it is not posible to characterize at least some of the pairwise farsightedly stable sets of networks.

Watts (2001) has analyzed the process of network formation in a dynamic framework where pairs of myopic agents meet and decide whether or not to form or sever links with each other based on the improvement the resulting network offers relative to the current network. An agent's payoff is determined as in Jackson and Wolinsky's (1996) connections model. It is shown that if the benefit from maintaining an indirect link is greater than the net benefit from maintaining a direct link, then it is difficult for the efficient network (the star network) to form. In fact, starting at the empty network, the efficient network only forms if the order in which the agents meet takes a particular pattern. Moreover, it is also shown that as the number of agents increases it becomes less likely that the efficient network forms. These results contrast with our result that, for such parameter values, every star network is a pairwise farsightedly stable set of networks whatever the number of farsighted agents. Hence, although myopic agents could form with a high probability inefficient networks, farsighted agents could always form the efficient network.

5 Buyer-seller networks

Corominas-Bosch (2004) has developped a simple model of trading networks with bilateral bargaining. The market consists of $\overline{s}$ sellers $1, 2, ..., \overline{s}$ and $\overline{b}$ buyers $\overline{s}+1, \overline{s}+2, ..., \overline{s}+\overline{b}$. We denote the set of buyers as B and the set of sellers as S. Each seller owns a single object to sell that has no value to the seller. Buyers have a valuation of 1 for an object and do not care from whom they purchase the good. If a seller and a buyer trade at price p, the seller receives a payoff of p and the buyer a payoff of $1-p$. Agents are embedded in a network that links sellers and buyers, and trade is only possible among linked agents. That is, a link in the network represents the opportunity for a buyer and a seller to bargain and potentially exchange an object.[9] Let $\mathbb{G}(S, B) = \{g \in \mathbb{G} \mid ij \in g \Leftrightarrow i \in S \text{ and } j \in B\}$ be the set of feasible buyer-seller networks. Agents incur a cost of maintaining each link equal to c_s for sellers and to c_b for buyers. So the payoff to an agent is her payoff from any trade on the network, less the cost of maintaining any links that she is involved with.

In the first period sellers simultaneously call out prices. A buyer can only select from the prices that she has heard called out by the sellers to whom she is linked. Buyers simultaneously respond by either choosing to accept a single price offer received or rejecting all price offers received.[10] At the end of the period, trades are made and buyers and sellers who have traded are cleared from the market. In the next period the situation reverses and buyers call out prices. These are then either accepted or rejected by the sellers connected to them. Each period the role of proposer and responder alternates and this process repeats itself until all remaining buyers and sellers are not linked to each other. Buyers and sellers are impatient so that a transaction at price p in period t is worth $\delta^t p$ to a seller and $\delta^t(1-p)$ to a buyer with $0 < \delta < 1$ being the common discount factor. In a subgame perfect equilibrium with very patient agents (δ close to 1), there are effectively three possible outcomes for any given agent (ignoring the costs of maintaining links): either he or she gets all the available gains from trade (1), or half of the gains from trade (1/2),

[9] A link is necessary between a buyer and a seller for a transaction to occur, but if an agent has several links, then there are several possible trading patterns. The network structure essentially determines the bargaining power of buyers and sellers.

[10] If there are several sellers who have called out the same price and/or several buyers who have accepted the same price, and there is any discretion under the given network connections as to which trades should occur, then there is a careful protocol for determining which trades occur. The protocol is essentially designed to maximize the number of transactions.

or none of the available gains from trade (0). Corominas-Bosch (2004) has provided an algorithm that subdivides any network into three types of subnetworks: those in which a set of sellers are collectively linked to a larger set of buyers (sellers obtain 1 as payoffs, and buyers receive 0); those in which the collective set of sellers is linked to the same-sized collective set of buyers (each receives 1/2); and those in which sellers outnumber buyers (sellers receive 0, and buyers get 1).[11]

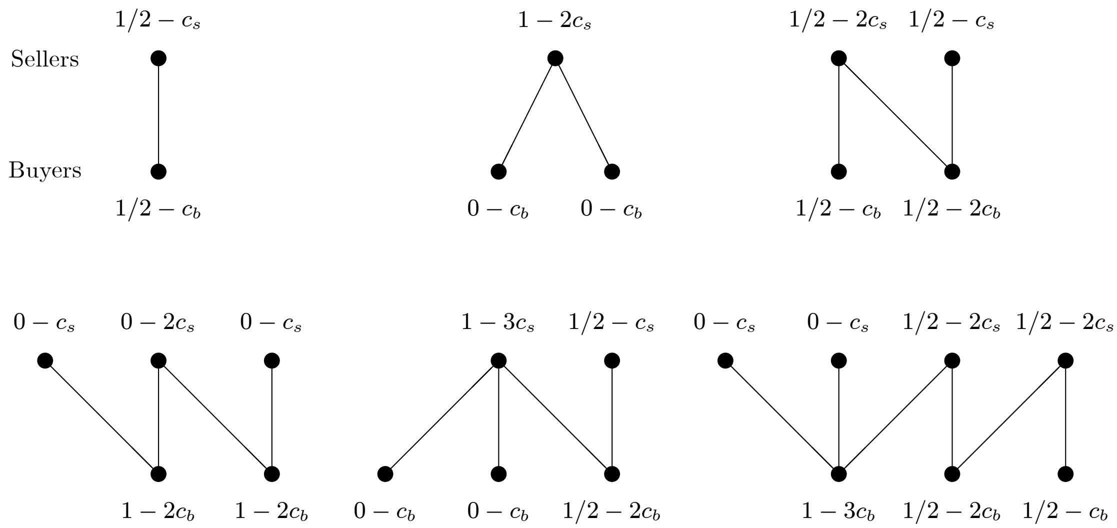

Figure 2 : Limit payoffs in the Corominas-Bosch (2004) model for some networks.

Let G_2 be the set of all buyer-seller networks consisting of pairs and so that the maximum number of potential pairs must form. That is, $G_2 = \{g \in \mathbb{G}(S, B) \mid l(g) = \min\{\#S, \#B\}$ and $l_i(g) \leq 1 \ \forall i \in S \cup B\}$ where $l(g)$ is the number of links in g and $l_i(g)$ is the number of links player i has in g.

Proposition 4 (Jackson, 2003) *In the Corominas-Bosch model with $1/2 > c_s > 0$ and $1/2 > c_b > 0$, the set of pairwise stable networks is G_2 which is exactly the set of strongly efficient networks.*

[11] The algorithm works as follows. Step 1a: Identify groups of two or more sellers who are all linked only to the same buyer. Regardless of that buyer's other connections, eliminate that set of sellers and buyer (with the buyer obtaining 1 and the sellers receiving 0). Step 1b: On the remaining network, repeat step 1a but with the role of buyers and sellers reversed. Step k: Proceed inductively in k, each time identifying subsets of at least k sellers who are collectively linked to some set of fewer-than-k buyers, or some collection of at least k buyers who are collectively linked to some set of fewer-than-k sellers. End: When all such subgraphs are removed, the buyers and sellers in the remaining network are such that every subset of sellers is linked to at least as many buyers and vice versa, and the buyers and sellers in that subnetwork get 1/2.

The intuition for this result is straightforward. An agent having a payoff of 0 cannot have any links since by deleting a link she could save the link cost and not lose any benefit. So, all agents who have links must obtain payoffs of 1/2 (ignoring the costs of maintaining links). Then, we can show that if there are extra links in such a network relative to the strongly efficient network which consists of a maximal number of disjoint linked pairs, some links could be deleted without changing the payoffs from trade but saving link costs. Thus, a pairwise stable network must consist of linked pairs, and the maximum number of potential pairs must form. Notice that if $1/2 < c_s$ and/or $1/2 < c_b$ then the empty network is the unique pariwise stable network. The empty network is strongly efficient only if $c_s + c_b \geq 1$.

Let $\overline{B} = \{\widetilde{B} \subseteq B \mid \#\widetilde{B} = \min\{\#S, \#B\}\}$ and $\overline{S} = \{\widetilde{S} \subseteq S \mid \#\widetilde{S} = \min\{\#S, \#B\}\}$. Given $\widetilde{B} \in \overline{B}$ and $\widetilde{S} \in \overline{S}$, let $G_2(\widetilde{B}, \widetilde{S}) = \{g \in \mathbb{G}(S, B) \mid l(g) = \min\{\#S, \#B\},\ l_i(g) = 1$ $\forall i \in \widetilde{S} \cup \widetilde{B}$, and $l_i(g) = 0\ \forall i \notin \widetilde{S} \cup \widetilde{B}\}$. Of course, $G_2(\widetilde{B}, \widetilde{S}) \subseteq G_2$.

Proposition 5 *In the Corominas-Bosch model with* $1/2 > c_s > 0$ *and* $1/2 > c_b > 0$*, for all* $\widetilde{B} \in \overline{B}$ *and* $\widetilde{S} \in \overline{S}$*, the set* $G_2(\widetilde{B}, \widetilde{S})$ *is a pairwise farsightedly stable set of networks.*

Proof. Take any $\widetilde{B} \in \overline{B}$ and $\widetilde{S} \in \overline{S}$. First, we show that for every $g' \notin G_2(\widetilde{B}, \widetilde{S})$ there is $g \in G_2(\widetilde{B}, \widetilde{S})$ such that $g \in F(g')$. Notice that, for every $g \in G_2(\widetilde{B}, \widetilde{S})$, each agent receives either $Y_i(g) = 1/2 - c_i > 0$ if agent i is linked to another agent or $Y_i(g) = 0$ if agent i has no link, and $Y_i(g_1) = Y_i(g_2)$ for all $g_1, g_2 \in G_2(\widetilde{B}, \widetilde{S})$, for all $i \in N$. Start with g' and build a sequence of networks where at each step some agent (who is looking forward to g) deletes a link until we reach a network g'' consisting only of linked pairs of agents and/or agents having no links. Then, agents successively add the links that belong to g but do not belong to g''. Finally, at each following step some agent who has two links at the current network, one link with her partner in g and one link with another partner, deletes the latter link until we reach the network g.

Step 1a: Agents who receive a payoff strictly less than 0 successively delete a link. Each agent is willing to delete a link looking forward to g since $Y_i(g) \geq 0$ for all $i \in S \cup B$. Step 1b: On the remaining network, delete a link from an agent who receives a payoff of $1/2 - l_i c_i$ with $l_i > 1$ and who obtains a payoff of $1/2 - c_i$ at the endpoint g. Step k: Proceed inductively in k, each time agents who receive a payoff strictly less than 0 successively delete a link; then, on the remaining network, delete a link from an agent who receives a payoff of $1/2 - l_i c_i$ with $l_i > 1$ and who obtains a payoff of $1/2 - c_i$ at the endpoint g. Step K: When all such links are removed, we end up at a network $g'' \in \{g \in \mathbb{G}(S, B) \mid l(g) \leq \min\{\#S, \#B\}$ and $l_i(g) \leq 1\ \forall i \in S \cup B\}$ where all the buyers and sellers in g'' that do have a link get a payoff of $1/2 - c_i$ while the others get 0. Select $g \in G_2(\widetilde{B}, \widetilde{S})$ such that $g \cap g'' \supseteq \widetilde{g} \cap g''$ for all $\widetilde{g} \in G_2(\widetilde{B}, \widetilde{S})$. Step $K+1$: Agents successively add the links that belong to g but do not belong to g''. That is, a pair of agents i and j will add the link ij so that $ij \in g$ and $ij \notin g''$. Since at least one of the agent has no link at g'', say agent i ($l_i(g'') = 0$), then $Y_i(g'') = 0 < Y_i(g) = 1/2 - c_i$, and so agent i is willing to add the link. The other agent (agent j) has either no link (which gives him a payoff of 0) or has one link (which gives him a payoff of $1/2 - c_j$) and so he agrees to add the link with agent j since $Y_j(g'') \leq Y_j(g)$. When all such links are added, we end up at a network g'''. Step $K+2$: Agents that have a link in g'' but do not have a link in g are linked in g''' to some agent who has two links in g''' and so obtain a payoff of $0 - c_i$. Those agents successively delete their links looking forward to g. When all such links are removed, we end up at the network g.

Second, we show that for every $g \in G_2(\widetilde{B}, \widetilde{S})$ we have that $F(g) \cap G_2(\widetilde{B}, \widetilde{S}) = \varnothing$. Since $Y_i(g_1) = Y_i(g_2)$ for all $g_1, g_2 \in G_2(\widetilde{B}, \widetilde{S})$ and for all $i \in S \cup B$, it follows that $g_1 \notin F(g_2)$ for all $g_1, g_2 \in G_2(\widetilde{B}, \widetilde{S})$. Theorem 3 in Herings, Mauleon and Vannetelbosch (2009) states that if for every $g' \in \mathbb{G} \setminus G$ we have $F(g') \cap G \neq \emptyset$ and for every $g \in G$, $F(g) \cap G = \emptyset$, then G is a pairwise farsightedly stable set. Hence, we have that $G_2(\widetilde{B}, \widetilde{S})$ is a pairwise farsightedly stable set. ■

Proposition 6 *In the Corominas-Bosch model with* $1/2 > c_s > 0$ *and* $1/2 > c_b > 0$*, there does not exist a pairwise farsightedly stable set* G *such that* $G \cap G_2 = \varnothing$*.*

Proof. We will show that for all $g' \notin G_2$ and for all $g \in G_2$ we have that $g' \notin F(g)$ which guarantees that there does not exist a pairwise farsightedly stable set G such that $G \cap G_2 = \varnothing$. The only networks $g' \notin G_2$ that some forward looking agents may prefer to $g \in G_2$ are such that the deviating agents obtain a payoff of 1 in g' (ignoring the costs of maintaining links). To obtain 1 the deviating agents will have to form links along the sequence with agents that will obtain 0 in g' (ignoring the costs of maintaining links). But, before forming these additional links with the original deviating agents, these agents have a payoff of either 1/2 or 0 (ignoring the costs of maintaining links), and thus, they have incentives to block the formation of any additional costly link. ■

In the bargaining model of Corominas-Bosch (2004) myopic or farsighted notions of stability sustain the set of strongly efficient networks when the costs of maintaining links are not too large. Notice that if $1/2 < c_s$ and/or $1/2 < c_b$ then a set consisting of the empty network is obviously the unique pariwise farsightedly stable set. In that case, on at least one side of the market (buyers or sellers) agents who have some link in any network receive a payoff strictly less than 0 and thus are willing to delete their links looking forward to the empty network. It also implies that there are no farsighted improving path emanating from the empty network.

The Kranton and Minehart (2001) model of buyer-seller networks is similar to the Corominas-Bosch model except that the valuations of the buyers for an object are random and the determination of prices is made through an auction rather than alternating-offer bargaining. Consider a version of the model with one seller ($\#S = 1$) and some potential buyers ($\#B \geq 1$). So, there is one seller who has an indivisible object for sale and b potential buyers who have utilities for the object, denoted u_i, which are uniformly and independently distributed on $[0, 1]$. The object to sell has no value to the seller. Each buyer knows his own valuation, but only the distribution over the buyers' valuations. The seller also knows only the distribution of buyers' valuations. The object is sold by means of a standard second-price auction. Only the buyers who are linked to the seller participate to the auction. The number of buyers linked to the seller is given by $l(g)$. For a cost per link of c_s to the seller and c_b to the buyer, the allocation rule for any network g with $l(g) \geq 1$ links between the buyers and the seller is

$$Y_i(g) = \begin{cases} \frac{1}{l(g)(l(g)+1)} - c_b & \text{if } i \text{ is a linked buyer} \\ \frac{l(g)-1}{l(g)+1} - l(g)c_s & \text{if } i \text{ is the seller} \\ 0 & \text{if } i \text{ is a buyer without any links.} \end{cases},$$

The value function is $v(g) = \frac{l(g)}{l(g)+1} - l(g)(c_s + c_b)$, which is simply the expected value of the object to the highest valued buyer less the cost of links. Let l_s^* be the number of links such that

$$\frac{2}{l\,(l+1)} \geq c_s \text{ and } \frac{2}{(l+1)\,(l+2)} < c_s,$$

which is the optimal number of links for the seller. Let l_b^* be the number of links such that

$$\frac{1}{l(l+1)} \geq c_b \text{ and } \frac{1}{(l+1)\,(l+2)} < c_b,$$

which is the maximal number of links up to which buyers make positive payoffs. A network g such that $l(g) = \min\{l_s^*, l_b^*\}$ is pairwise stable. Notice that if $\frac{2}{l_s^*(l_s^*+1)} = c_s$, $\frac{1}{l_b^*(l_b^*+1)} = c_b$ and $l_s^* = l_b^*$ then $g - ij$ such that $l(g) = \min\{l_s^*, l_b^*\}$ is pairwise stable too.

Proposition 7 *In the Kranton and Minehart model with one seller,*

(i) *If* $\frac{2}{l_s^*(l_s^*+1)} = c_s$, $\frac{1}{l_b^*(l_b^*+1)} = c_b$ *and* $l_s^* = l_b^*$ *then* $G_{-1} = \{g \in \mathbb{G}(\{1\}, B) \mid l(g) \in \{l_s^* - 1, l_s^*\}\}$ *is the unique pairwise farsightedly stable set.*

(ii) *Otherwise,* $\{g\}$ *with* $g \in G_1 = \{g \in \mathbb{G}(\{1\}, B) \mid l(g) = \min\{l_s^*, l_b^*\}\}$ *are the unique pairwise farsightedly stable sets.*

Proof.

(i) Suppose $\frac{2}{l_s^*(l_s^*+1)} = c_s$, $\frac{1}{l_b^*(l_b^*+1)} = c_b$ and $l_s^* = l_b^*$. Let $G_{-1} = \{g \in \mathbb{G}(\{1\}, B) \mid l(g) \in \{l_s^* - 1, l_s^*\}\}$. We have $Y_s(g) = Y_s(g')$ for all $g, g' \in G_{-1}$, $Y_i(g) = 0$ for all $g \in G_{-1}$ such that $l(g) = l_s^*$, $i \in B$, and $Y_i(g) = 0$ for all $g \in G_{-1}$ such that $l(g) = l_s^* - 1$, $i \in B$ with $l_i(g) = 0$. It follows that (a) $g' \notin F(g)$ for all $g, g' \in G_{-1}$; (b) for all $g' \notin G_{-1}$ there is $g \in F(g')$ such that $g \in G_1$; (c) $g' \notin F(g)$ for all $g \in G_{-1}$, $g' \notin G_{-1}$. Thus, G_{-1} is the unique pairwise farsightedly stable set.

(ii) Suppose $\frac{2}{l_s^*(l_s^*+1)} > c_s$ and/or $\frac{1}{l_b^*(l_b^*+1)} > c_b$ and/or $l_s^* \neq l_b^*$; and let $G_1 = \{g \in \mathbb{G}(\{1\}, B) \mid l(g) = \min\{l_s^*, l_b^*\}\}$. It is quite straighforward that (a) $g' \notin F(g)$ for all $g' \notin G_1$ and $g \in G_1$; (b) $g' \in F(g)$ for all $g, g' \in G_1$; (c) $g \in F(g')$ for all $g \in G_1$, $g' \notin G_1$. Then, it follows that $\{g\}$ with $g \in G_1$ are the unique pairwise farsightedly stable sets. ∎

In the Kranton and Minehart (2001) model it is possible for non-trivial pairwise stable networks to be Pareto inefficient. Consider a population with two sellers and four buyers. Let agents 1 and 2 be the sellers and 3, 4, 5 and 6 be the buyers. Some straightforward but tedious calculations lead to the payoffs to the agents which are given in Figure 3 and Figure 4 for selected networks. For instance, when $c_s = 5/60$ and $c_b = 1/60$, there are three types of pairwise stable networks: the empty network, networks that look like $\{13, 14, 15, 16\}$, and networks that look like $\{13, 14, 15, 24, 25, 26\}$. Both the empty network and $\{13, 14, 15, 24, 25, 26\}$ are not Pareto efficient, while $\{13, 14, 15, 16\}$ is. The empty network and the network $\{13, 14, 15, 24, 25, 26\}$ are Pareto dominated by $\{13, 14, 25, 26\}$. In addition, the network $\{13, 14, 15, 16\}$ is not efficient. The network $\{13, 14, 25, 26\}$ is efficient but is not pairwise stable since agents 1 and 5 have incentives to add a link. However, the set consisting of $\{13, 14, 25, 26\}$ is a pairwise farsightedly stable set.

One open question is whether Pareto inefficient networks could belong to some pairwise farsightedly stable set with many sellers and buyers.

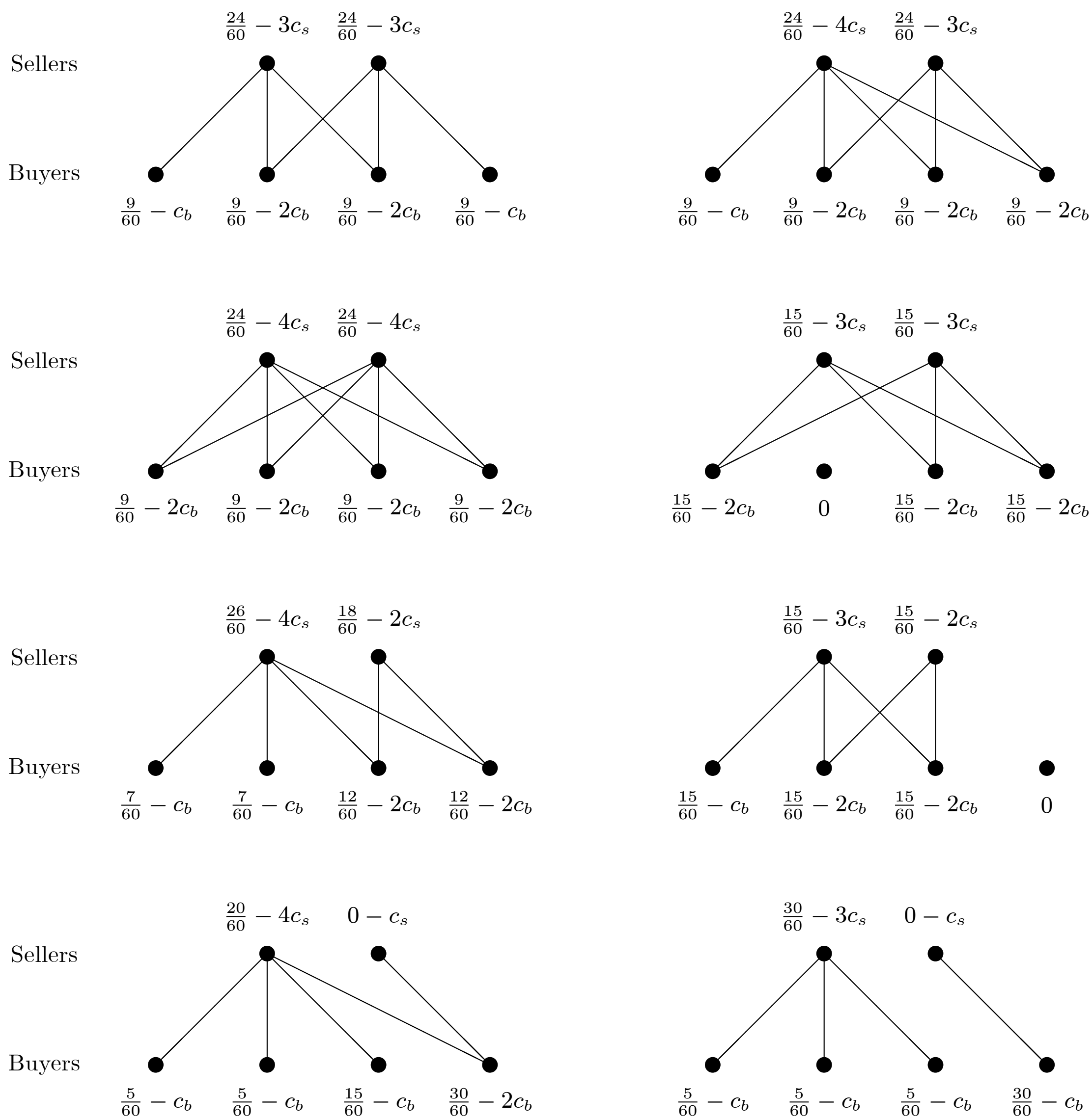

Figure 3 : Payoffs in the Kranton and Minehart (2001) model for selected networks.

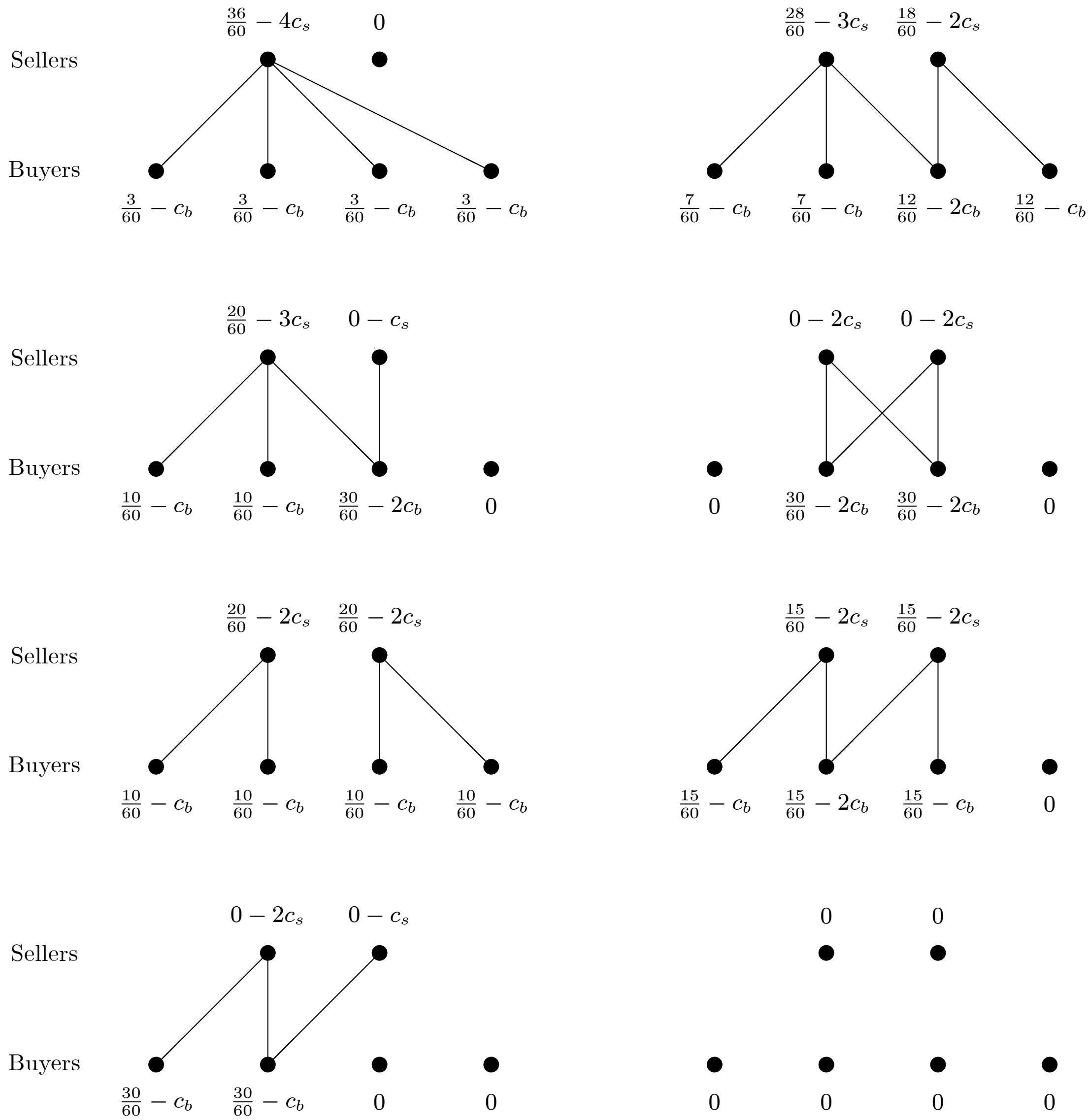

Figure 4 : Payoffs in the Kranton and Minehart (2001) model for selected networks (continued).

6 Farsighted stability and efficiency

Herings, Mauleon and Vannetelbosch (2009) have shown that the set of pairwise farsightedly stable networks and the set of strongly efficient networks, those which are socially optimal, may be disjoint for all allocation rules that are component balanced and anonymous. However, as already mentioned, if there is a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique pairwise farsightedly stable set. Suppose that Y is the egalitarian allocation rule and $E(v)$ is the set of strongly efficient networks. Then, $E(v)$ is the unique pairwise farsightedly stable set.

Let us introduce now a notion of pairwise farsighted stability that only accounts for deviations that make all players strictly better off.

Definition 3 *A strict farsighted improving path from a network* g *to a network* $g' \neq g$ *is a finite sequence of networks* $g_1, \ldots, g_K$ *with* $g_1 = g$ *and* $g_K = g'$ *such that for any* $k \in \{1, \ldots, K-1\}$ *either:*

(i) $g_{k+1} = g_k - ij$ *for some* ij *such that* $Y_i(g_K, v) > Y_i(g_k, v)$ *or* $Y_j(g_K, v) > Y_j(g_k, v)$*, or*

(ii) $g_{k+1} = g_k + ij$ *for some* ij *such that* $Y_i(g_K, v) > Y_i(g_k, v)$ *and* $Y_j(g_K, v) > Y_j(g_k, v)$*.*

For a given network g, let $F^s(g)$ be the set of networks that can be reached by a strict farsighted improving path from g. We have that $F^s(g) \subseteq F(g)$.

Definition 4 *A set of networks* $G \subseteq \mathbb{G}$ *is a strict pairwise farsightedly stable set with respect* v *and* Y *if*

(i) $\forall\ g \in G$*,*

- **(ia)** $\forall\ ij \notin g$ *such that* $g + ij \notin G$*,* $\exists g' \in F^s(g + ij) \cap G$ *such that* $Y_i(g', v) \leq Y_i(g, v)$ *or* $Y_j(g', v) \leq Y_j(g, v)$*,*
- **(ib)** $\forall\ ij \in g$ *such that* $g - ij \notin G$*,* $\exists g', g'' \in F^s(g - ij) \cap G$ *such that* $Y_i(g', v) \leq Y_i(g, v)$ *and* $Y_j(g'', v) \leq Y_j(g, v)$*,*

(ii) $\forall g' \in \mathbb{G} \setminus G$*,* $F^s(g') \cap G \neq \emptyset$*.*

(iii) $\nexists\ G' \subsetneq G$ *such that* G' *satisfies Conditions (ia), (ib), and (ii).*

It is straightforward that if $\{g\}$ is a strict pairwise farsightedly stable set then $\{g\}$ is a pairwise farsightedly stable set. However, the reverse is not true.

Let g^S be the collection of all subsets of $S \subseteq N$ with cardinality 2. Let

$$g(v, S) = \underset{g \subseteq g^S}{\operatorname{argmax}} \frac{v(g)}{\#N(g)}$$

be the network with the highest per capita value out of those that can be formed by players in $S \subseteq N$. Given a component additive value function v, find a network g^v through the following algorithm. Pick some $h_1 \in g(v, N)$. Next, pick some $h_2 \in g(v, N \setminus N(h_1))$. At stage k pick some $h_k \in g(v, N \setminus \cup_{i \leq k-1} N(h_i))$. Since N is finite this process stops after a finite number K of stages. The union of the components picked in this way defines a network g^v. We denote by G^v the set of all networks that can be found through this algorithm.[12] More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

[12] This algorithm was first introduced by Banerjee (1999) who works with a notion of strong stability but one that only accounts for deviations that make all players strictly better off.

Proposition 8 *Consider any anonymous and component additive value function v. The set G^v is the unique strict pairwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. Consider any anonymous and component additive value function v. First we show that $F^s(g) = \varnothing$ for all $g \in G^v$ under the componentwise egalitarian allocation rule Y^{ce}. Take any $g \in G^v$. Players belonging to $N(h_1)$ in g who are looking forward will never engage in a move since they can never be strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}. Players belonging to $N(h_2)$ in g who are looking forward will never engage in a move since the only possibility to obtain a strictly higher payoff is to end up in h_1 (if h_1 gives a strictly higher payoff than h_2) but players belonging to $N(h_1)$ will never engage a move. So, players belonging to $N(h_2)$ can never end up strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}. Players belonging to $N(h_k)$ in g who are looking forward will never engage in a move since the only possibility to obtain a strictly higher payoff is to end up in h_1 or h_2 ... or h_{k-1} but players belonging to $\cup_{i \leq k-1} N(h_i)$ will never engage a move. So, players belonging $N(h_k)$ can never end up strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}; and so on. Thus, $F^s(g) = \varnothing$.

Second, we show in a constructive way that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in F^s(g')$ under the componentwise egalitarian allocation rule Y^{ce}. Take any $g' \notin G^v$ and $g \in G^v$. In g' all players are strictly worse off than the players belonging to $N(h_1)$ in g under the componentwise egalitarian allocation rule Y^{ce}. From g', let the players who belong to $N(h_1)$ in g and are looking forward to g first deleting successively all their links and then building successively the links in h_1 (leading to $g'' = g' + h_1 - \{ij \mid i \in N(h_1)\}$). Along the sequence from g' to g'' all players who are moving always strictly prefer the end network g to the current network. Once g'' (and h_1) is formed, all the remaining players who are belonging to $N \setminus N(h_1)$ in g'' are strictly worse off than the players belonging to $N(h_2)$ in g. From g'', let the players who belong to $N(h_2)$ in g and who are looking forward to g first deleting successively all their links and then building successively the links in h_2 (leading to $g''' = g' + h_1 + h_2 - \{ij \mid i \in N(h_1) \cup N(h_2)\}$); and so on until we reach the network g. Thus, we have build a strict farsighted improving from g' to g; $g \in F^s(g')$.

Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique (strict) pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F^s(g) = \varnothing\}$ and for every $g' \in \mathbb{G} \setminus G$, $F^s(g') \cap G \neq \varnothing$, we have that G^v is the unique strict pairwise farsightedly stable set. ■

A network g is a strict pairwise stable network with respect to value function v and allocation rule Y if (i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$, and (ii) for all $ij \notin g$, if $Y_i(g,v) < Y_i(g+ij,v)$ then $Y_j(g,v) \geq Y_j(g+ij,v)$. We have that all networks belonging to G^v are strict pairwise stable networks. So, strict pairwise farsighted stability refines the notion of strict pairwise stability under Y^{ce}. However, this proposition does not hold under the notion of (weak) pairwise farsighted stability. Consider a situation with five players where the payoffs to players in networks of the types $g^c = \{12, 23, 45\}$ and $g^d = \{12, 45\}$ are, respectively, $Y_1(g^c) = Y_2(g^c) = Y_3(g^c) = Y_4(g^c) = Y_5(g^c) = 10$ and $Y_1(g^d) = Y_2(g^d) = Y_4(g^d) = Y_5(g^d) = 10, Y_3(g^d) = 0$ (see right part of Figure 5), while in all other networks payoffs are equal to zero. Under the above algorithm, G^v consists of all networks of the types g^c and g^d, but there is a (weak) farsighted improving path from g^d to g^c. Using Jackson's algorithm would not help in recovering the proposition.[13] For instance, consider a situation with six players where the payoffs to players in networks of the types $g^a = \{12, 23, 45, 56\}$ and $g^b = \{12, 34, 56\}$ are equal to 10 (see left part of Figure 5), while in all other networks payoffs are equal to zero. Jackson's algorithm would only select the networks of the type g^a while there are no farsighted improving path from g^b to g^a and vice-versa.

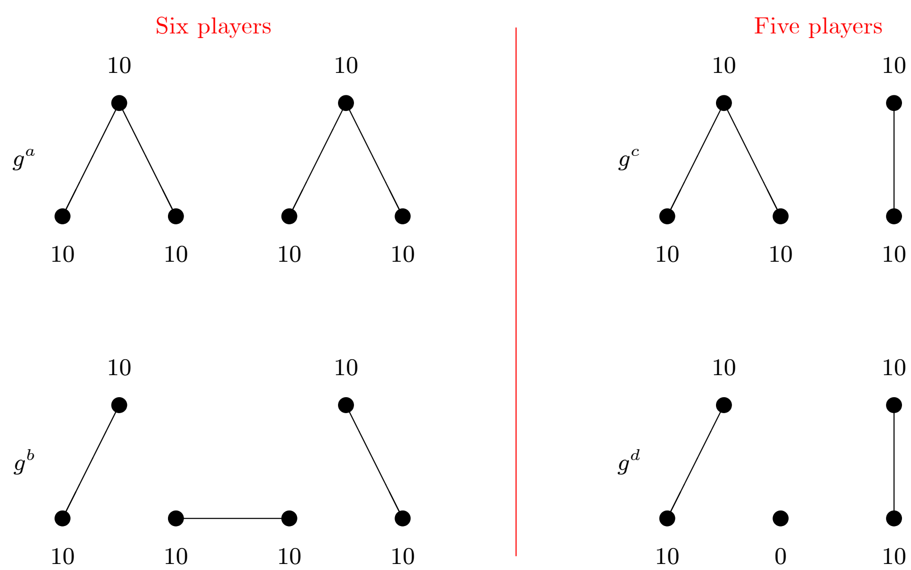

Figure 5 : Strict versus weak pairwise farsighted stability.

A value function v is *top convex* if some strongly efficient network also maximizes the

[13] Jackson (2005) has proposed an alternative algorithm which is a bit different since it requires to pick the maximal number of links in the definition of each h_k. Under a component additive v, a network defined by Jackson's algorithm is a pairwise stable and Pareto efficient network under the componentwise egalitarian allocation rule Y^{ce}.

per capita value among players. Let $\rho(v,S) = \max_{g \subseteq g^S} v(g)/\#S$. The value function v is *top convex* if $\rho(v,N) \geq \rho(v,S)$ for all $S \subseteq N$.

Proposition 9 *Consider any anonymous and component additive value function v. The set of strongly efficient networks $E(v)$ is the unique pairwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce} if and only if v is top convex.*

Proof. Consider any anonymous and component additive value function v. ($\Leftarrow$) Top convexity implies that all components of a strongly efficient network must lead to the same per-capita value (if some component led to a lower per capita value than the average, then another component would have to lead to a higher per capita value than the average which would contradict top convexity). It follows that under the componentwise egalitarian allocation rule any $g \in E(v)$ Pareto dominates all $g' \notin E(v)$. Then, it is immediate that $g \in F(g')$ for all $g' \in \mathbb{G} \setminus E$ and that $F(g) = \varnothing$. Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \varnothing\}$ and for every $g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \varnothing$, we have that $E(v)$ is the unique pairwise farsightedly stable set.

($\Rightarrow$) Since $E(v)$ is the unique pairwise farsightedly stable set, we have $F(g) = \varnothing$ for all $g \in E(v)$. It follows that under the componentwise egalitarian allocation rule (i) $Y_i^{ce}(g,v) = Y_j^{ce}(g,v) = Y_i^{ce}(g',v) = Y_j^{ce}(g',v)$ for all $i,j \in N$ and for all $g,g' \in E(v)$; (ii) $Y_i^{ce}(g,v) \geq Y_i^{ce}(g',v)$ for all $i \in N$, for all $g \in E(v)$, for all $g' \notin E(v)$. Thus, v is top convex. ■

Jackson and van den Nouweland (2005) have shown that the set of strongly efficient networks coincides with the set of strongly stable networks under the componentwise egalitarian allocation rule if and only if v is top convex.[14] Hence, the set of strongly stable networks is the unique pairwise farsightedly stable set under the componentwise egalitarian allocation rule if and only if the value function is top convex. So, pairwise farsighted stability selects under Y^{ce} the pairwise stable networks that are immune to coalitional deviations if and only if v is top convex. Note that top convexity is a condition that is satisfied in some natural situations. For instance, the value function of the symmetric connections model is top convex for all values of $\delta \in [0,1)$ and $c \geq 0$, so that all strongly

[14] Jackson and van den Nouweland (2005) have proposed a refinement of pairwise stability where coalitionwise deviations are allowed: the strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component.

efficient networks with respect to v form the unique pairwise farsightedly stable set with respect to Y^{ce} and v.

Finally, let us turn to the notion of strict pairwise farsighted stability. Consider a situation with five players where the payoffs to players in networks of the type $g^a = \{12, 23, 45\}$ are $Y_1(g^a) = Y_2(g^a) = Y_3(g^a) = 10$, $Y_4(g^a) = Y_5(g^a) = 5$ while in all other networks payoffs are equal to zero. The set of strongly efficient networks consists of networks of the type g^a and is the unique strict pairwise farsightedly stable set. However, v does not satisfy top convexity. Thus, under the notion of strict pairwise farsighted stability, top convexity is not necessary to sustain the set of strongly efficient networks as the unique pairwise farsightedly stable set.

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