Contracts with aftermarkets

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Abstract

I merge the standard Principal Agent model with a CAPM-type financial market, to study the interactions of contracts and financial markets. I prove existence of equilibrium in this model. I prove a Revelation Principle type of result to restrict the study of firm design to that of simple sharing rules. I study economies for which markets have an insurance effect on compensation contracts. I show sufficient conditions for lower variance to obtain. In this context I show the effect of markets' size on efficiency.

1 Introduction

I study workers' compensation and financial markets in a unified framework.

The size of compensation contracts in US amounts to more than 60% of US GDP. This makes compensation a relevant determinant of the payoff structure of assets issued on the market.

It is not clear that compensation packages are independent from the type of market (if any) on which firms trade their profits. What type of contracts does a non-listed company offer to its employees? How does it compare to those observed in Wall Street traded companies? Are the contracts different in firms controlled by Private Equity groups?

The answers to these questions are relevant for companies, and also for policy-makers, who write governance and disclosure rules. Many companies in developing countries are not listed, and an increasing number of companies go out of the market into private equity control and then back. [1]

[1] For current policy discussion in the field, see for example the Corporate Affairs section of the OECD website.

To study these problems, I integrate two popular models. Principal Agent Models are the main theoretical tool used to study compensation schemes. Optimal incentive contracts are derived, analyzed, and sometimes tested against available data. Asset pricing models have been developed to study the efficiency of markets as means to risk sharing, aggregating information, and inducing Pareto Efficient allocations. Perhaps the most used asset market model is the Capital Asset Pricing Model, CAPM.

I analyze a Principal-Agent model, with many pairs of principals and agents, where Principals have access to an asset market after the contracting stage.

- The contracting procedure is the standard one: principals make a take-it-or-leave-it offers to randomly drawn agents. These offers take the form of a menu of contracts.
- The asset market is an Arrow-Debreu market where individuals have mean variance preferences, and a riskless asset is available, a CAPM economy.

The main idea in my model is that when Principals' create a menu of contracts to offer to an Agent, they are also designing securities to trade in the subsequent asset market. I study here a simple information structure where information is symmetric at the market stage, and Arrow-Debreu Security Market Equilibrium can be used as a solution concept for the last stage.

There are two important questions I try to address:

- How does the existence of asset markets affect the solution to the Principal-Agent problem?
- How does contracting affect the market portfolio and systematic risk?

The answer to the first question is not trivial, and it depends on preferences and on the distribution of returns of companies. The market provides provide an insurance or diversification effect to risk averse principals, allowing them to reach a higher utility and, in some cases, contracts will be influenced. I show that with symmetric information and independent returns the contracts offered are always safer.

The answer to the second question is simpler: the asymmetric information at the contracting stage imposes constraints on the security a principal can issue on the market. With a take-it-or-leave-it offer structure and complete information, every agent is pushed to their reservation utility, and principals trade the remainder. However, when agents' types are unknown, some Incentive Compatibility constraint is likely to be binding. This will change the securities which are issued, and hence the aggregate risk in the economy.

Section 2 gives an overview of existing literature on the interactions between contracts and asset markets. Section 3 provides a brief overview of the two benchmark models. In Section 4 I present a simple example to introduce the model. In Section 5 I formally characterize the economy in its primitives, information structures and action spaces. In Section 6 I describe and define Equilibrium. In Section 7 I provide a revelation principle result to reduce the dimensionality of the strategy space. In Section 8 I prove existence. Section 9 gives a sufficient condition for the insurance effect of markets to influence contracts.

2 Related Literature

There is an extensive literature on compensation and asset prices, but only a few papers look at both in equilibrium.

2.1 Interaction with Asset Prices

Several authors look at asset pricing in a context of delegated investment (For a survey, see Stracca, 2003). The effects on prices and returns of the classical informational asymmetries phenomena, moral hazard and adverse selection are studied in a CAPM setting[2], where a representative principal delegates his investing decisions to an agent. In this literature inefficiencies take the form of deviations from the non-delegated case equilibrium (asset prices, optimal portfolio composition and return). The typical framework with a representative investor, coupled with managers' abilities and actions having effect only on portfolios, is powerful but limiting in terms of analyzing the efficiency of the resulting allocations. An example of the questions typically asked in the finance literature can be found in Ou-Yang (2005). Taking managers' assignments as exogenous, but their actions to be unobserved, he tries general and specific (in terms of utility functions or risk aversion of agents) approaches to find how returns and prices are affected by moral hazard and finds that the CAPM β relation between returns still holds. His approach is powerful but more geared towards the effect of contracting on market prices. I am studying a situation of Hidden Type, and my results are focused on the effect of markets on contracts.

[2]Other authors build their model around APT

2.2 General Equilibrium Models

A few papers in general equilibrium are related in different ways to the problem at hand. Rahman (2005a, 2005b), Ellickson, Grodahl, Scotchmer and Zame (1999,2001,2005), Zame (2005) separately study economies where agents form team or clubs, which can have different functions such as production or consumption. Market clearing happens through pricing of payoff splitting agreements. These are Walrasian models of competitive contracting, but they do not include asset markets. Magill and Quinzii (2005) are specifically interested in management compensation under uncertainty, but the assignment of managers to firms is given and known at the time of contracting.

3 A Brief Reminder

I will first briefly introduce outlines of the two models I am working with.

3.1 CAPM

The Capital Asset Pricing Model has a long and non-linear history. As a result the term is used by different people with different connotations. The dominant use of the term at this day is to identify a class of financial market economies in which equilibria satisfy certain properties:

- *Beta-Pricing Relation* In equilibrium the price of any asset is expressed as a function of mean returns and their covariance with the market portfolio.
- *Portfolio Separation* In equilibrium every trader holds the same portfolio of risky securities, and different positions on the riskless asset.

The latter is exploited here because it implies analytical tractability:

In the original incarnation of the model this property naturally followed by the preferences of agents, by the existence of a riskless asset (and naturally no constraints on short sales), and distributional assumptions on returns. Through the years a consistent amount of research has been undertaken to extend the type of economies for which these propreties hold. As my main interest lies elsewhere I will make these same assumptions for the sake of tractability.

3.1.1 Primitives

In this economy *individuals* trade *financial assets.*

- There are N individuals, each individual i cares only about the mean and variance of returns of the portfolio held. Their attitude towards risk is summarized by the variance aversion parameter a and their utility function is given by
$$U(\mu, \sigma^2) = \mu - \frac{a}{2}\sigma^2$$
- There are 1 riskless asset and K risky asset. Because individuals care about mean and variance and I will allow only for assets fully characterized by mean and variance, The only relevant objects are a K-dimensional vector of returns $\tilde{\mu}$ and the $K \times K$ variance-covariance matrix $\mathbf{\Omega}$.

  In the form
$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_P \end{bmatrix}, \Omega = \begin{bmatrix} \sigma_1^2 & \dots & \rho_{1P}\sigma_1\sigma_P \\ \vdots & \ddots & \vdots \\ \rho_{1P}\sigma_1\sigma_P & \dots & \sigma_P^2 \end{bmatrix}$$
  The expression $\rho_{ij}\sigma_i\sigma_j$ identifies the covariance between the returns of asset i and asset j

3.1.2 Equilibrium

After agents trade they will all end up with an identical portfolio of risky securities and different quantities of the riskless asset. The portfolio of risky securities held by each agent is going to be one N-th of the aggregate endowment of securities.

If w_i is the wealth of agent i he will hold $\frac{1}{N}$ of the market portfolio and $w_i - \frac{\sum_{j=1}^{N}}{N} w_j$ units of riskless asset

3.2 P-A

I will describe a very synthetic model of a firm and highlight the relevant issues. One individual, the *principal* p owns a technology, but is not going to work on it. She will rather hire another individual, the *agent* a. p does not know a's skills, but she has the opportunity to design a *menu* of *contracts* and offer it to the agent for him to pick one.

The technology will produce a positive real number of widgets, and a contract specifies compensation in the form of a linear function of the yield X: $\alpha + \beta X$.

Suppose the agent could have two types of skills $\{G, B\}$, and an agent can have each skill with probabilities q and $1 - q$. The yield of the technology is a positive random variable distributed according to the agent's skills X_G or

X_B. When p designs a menu, she will write two contracts, one for each type of agent, and she will have to make sure, that

IR The proposed compensation makes work worthier than staying at home (*Individual Rationality*).

IC An agent of type G will pick the contract intended for him over the one intended for type B, and vice versa (*Incentive Compatibility*).

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The *Principal-Agent problem* in this case boils down to:

$$\max_{(\alpha_G,\beta_G),(\alpha_B,\beta_B)} qEU\left(-\alpha_G + (1-\beta_G)X_G\right) + (1-q)EU\left(-\alpha_B + (1-\beta_B)X_B\right)$$

$$\text{such that } EU\left(\alpha_G + \beta_G X_G\right) \geq \overline{u} \quad \text{(IR G)}$$

$$EU\left(\alpha_B + \beta_B X_B\right) \geq \overline{u} \quad \text{(IR B)}$$

$$EU\left(\alpha_G + \beta_G X_G\right) \geq EU\left(\alpha_B + \beta_B X_G\right) \quad \text{(IC G)}$$

$$EU\left(\alpha_B + \beta_B X_B\right) \geq EU\left(\alpha_G + \beta_G X_B\right) \quad \text{(IC B)}$$

Where $\overline{u}$ is the reservation utility of agents (ie: what they would get if they "stayed at home"). The first two constraints are the IR constraints, and the others are the IC constraints.

4 A Simple Example

In this example, I show how markets can affect contracts in a very simple setting. Markets provide diversification to principals, this diversification opportunity makes sure that Principals will insure agents more than in a standard P-A model.

There are four individuals, with identical preferences over random variables, $U(X) = \mu_X - \frac{a}{2}\sigma_X^2$. Two of them, the Principals own an identical technology, which will return either 0 or 1. Two of them, the Agents, have the skills to operate the technology. Their skills are private information at the contracting stage. Both agents have the same reservation utility of $\frac{1}{3}$.

The skills of agents are identified with the probabilities of returns being 1, and they are $t_H = \frac{2}{3}$ for agent H and $t_L = \frac{1}{3}$ for agent L. The performance of one agent is stochastically independent from the performance of the other. The mean returns for an agent is given by $\mu_i = t_i$ and the variance is $\sigma_i^2 = t_i(1-t_i)$.

Each principal designs a menu of two linear contracts to offer to the agent he is going to be randomly matched with. A linear contract is a function of the form $y = \alpha + \beta x$, and is characterized by a pair (α, β) Principals will receive $y_p = -\alpha + (1-\beta)x$, the agent will receive $y_a = \alpha + \beta x$.

4.1 Standard Principal-Agent Model: No Market

The problem of principals is going to be:

$$\max_{\alpha_H,\beta_H,\alpha_L,\beta_L} \frac{1}{2}\left(\alpha_H + \beta_H t_H - \frac{a}{2}\beta_H^2\sigma_H^2\right) + \frac{1}{2}\left(\alpha_L + \beta_L \mu_L - \frac{a}{2}\beta_L^2\sigma_L^2\right)$$
$$\text{subject to} \quad IR_H, IC_H, IR_L, IC_L$$

The solution to a standard P-A model with linear contracts is to offer a menu of these two contracts:

$$y_L = \frac{1}{3}$$
$$y_H = \frac{1}{162} + \frac{1}{2}x$$

4.2 Principal-Agent with Financial Markets

Now suppose that the Principals can trade their claims on the asset market. The objective function of principals is now different, because it includes the outcome of markets. With mean-variance preferences the asset market equilibrium is determined by a few simple equations, which can be substituted in the objective function.

The outcome of the CAPM market:

- The market portfolio will be characterized by mean and variance
  - $\mu_{MKT}(\alpha_H, \beta_H, \alpha_L, \beta_L) = \alpha_L + \alpha_H + \beta_L t_L + \beta_H t_H$
  - $\sigma^2_{MKT}(\alpha_H, \beta_H, \alpha_L, \beta_L) = \beta_H^2 t_H(1-t_H) + \beta_L^2 t_L(1-t_L)$
- Equilibrium shares will be
  - Market Portfolio:
    $\theta_H = \theta_L = \frac{1}{2}$
  - Riskless asset:
    $\theta_i^P = (-\alpha_i + (1-\beta_i)t_i - \frac{a}{2}(1-\beta_H)^2\sigma_H^2 - \frac{-\alpha_H + (1-\beta_H)t_H - \frac{a}{2}(1-\beta_H)^2\sigma_H^2 - \alpha_L + (1-\beta_L)t_L - \frac{a}{2}(1-\beta_L)^2\sigma_L^2}{2}$

The Principals' problem when Markets are available will be:

$$\max_{\alpha_H,\beta_H,\alpha_L,\beta_L} \frac{1}{2}[(q_H - \frac{q_H+q_L}{2}) + \frac{1}{2}\mu_{MKT}(\alpha_H,\beta_H,\overline{\alpha}_L,\overline{\beta}_L)$$

$$-\frac{a}{2}\frac{1}{4}\sigma^2_{MKT}(\alpha_H,\beta_H,,\overline{\alpha}_L,\overline{\beta}_L)]$$

$$+\frac{1}{2}[(q_L - \frac{q_H+q_L}{2}) + \frac{1}{2}\mu_{MKT}(\overline{\alpha}_H,\overline{\beta}_H,\alpha_L,\beta_L)$$

$$-\frac{a}{2}\frac{1}{4}\sigma^2_{MKT}(\overline{\alpha}_H,\overline{\beta}_H,\alpha_L,\beta_L)]$$

$$\text{subject to}$$

$$IR_H, IC_H, IR_L, IC_L$$

The optimal contracts in this setting will be

$$y_L^{MKT} = \frac{1}{3}$$

$$y_H^{MKT} = \frac{23}{441} + \frac{3}{7}x$$

An important feature exhibited by this example is that optimal contracts are different when asset markets are present: they are more similar to wages. In this example the returns are independent, so that the market offers diversification opportunities, but not insurance. Diversification is however sufficient for principals to offer safer contracts to agents and achieve a higher expected utility.

5 Contracts AND Markets

make better My aim here is studying the interactions of contracting inside the firm with asset trading on financial markets. For this reason I consider a model where principals are allowed to trade their claim to profits on a market after contracting has taken place.

5.1 Primitives

There is a finite set of skills T. A population I of Principals and Agents $I = P \cup A$, with Mean Variance Preferences over random variables in the form

$E(X) - \frac{a}{2}Var(X)$. Agents $a \in A$ are endowed with their work skills $t \in T$, and a reservation utility $\overline{u}_a$

A finite set of technologies K, whose distribution of returns depend also on the skills of the agent operating them. I will consider here binary returns, which will be uniquely described by the vector of mean payoffs μ and the variance-covariance matrix Ω. Let $t(p)$ be the type of agent operating in principal's p firm.

$$\mu = \begin{bmatrix} \mu_1(t(1)) \\ \vdots \\ \mu_P(t(P)) \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \sigma_1^2(t(1)) & \dots & \rho_{1P}(t(1), t(P))\sigma_1(t(1))\sigma_P(t(P)) \\ \vdots & \ddots & \vdots \\ \rho_{1P}(t(1), t(P))\sigma_1(t(1))\sigma_P(t(P)) & \dots & \sigma_P^2(t(P)) \end{bmatrix}$$

Principals $p \in P$ are endowed with one unit of capital necessary to operate a technology $k \in K$[3]. Principals can be identified by their technology. There is also a riskless asset L, returning 1 unit of good in each and every state of the world.

To identify agents with their relevant characteristics consider a mapping *type* $: I \to T \cup K$, determining the type of each individual in the economy, and its restrictions.

$$t : A \to T$$
$$k : P \to K$$

All technologies in K require exactly one Principal and one Agent to be operated. A *contract* is a contingent agreement on how to split returns between the Principal and Agent forming a firm with technology k. I impose the restriction that these sharing rules be affine: if X is the random variable describing the profits of the firm, an admissible rule describing the principal's and agent's share must be of the form:

$$X_p = -\alpha + (1 - \beta)X$$
$$X_a = \alpha + \beta X$$

In this binary setting, affine sharing rules (more commonly called linear contracts), are exhaustive of all possible linear rules.

[3]In this model capital is non-homogenous: principals are endowed with a technology and cannot choose to invest capital in another one.

I want to look at a case where the type of a principal is public but the type of an agent is private information. The distribution of skills is common knowledge. Let $\mathcal{F}$ be the σ-field generated by the sets of k and t of principals and agents of each characteristic k and t $P_1, ..., P_{|K|}, A_1, ..., A_{|T|}$. The population measure ν defined on $(I, \mathcal{F})$ is common knowledge. This measure gives the proportion of agents of each type in the population. The distribution of types is $\lambda(A_t) = \frac{\nu_{|A}(A_t)}{\nu(a)}$

5.2 Timeline

I will call the primitives of the game that are common knowledge at every stage CMN, and they are:

- The vector of means and the variance-covariance matrix of payoffs as a function of the type of agents working in each firm.

$$\vec{\mu}(t(1), ...t(p))$$
$$\mathbf{\Omega}(t(1), ...t(p))$$

- The technologies K, the technology of each principal (the function $k(\cdot)$), the types T, and their distribution across the population ν.
- The variance aversion parameters a and their reservation utility $\overline{u}$.

As noted, an Agents' skills $t(\cdot)$ are private information and the realized matching is unknown. The economy reaches its equilibrium in 4 stages

1. Principals and agents are matched into pairs *randomly* and *simultaneously.*

   Nature randomly draws a matching τ, the probability of each matching is equal to $\frac{1}{P!}$.

   A matching is a bijection

$$\begin{aligned}
&\tau : I \to I \\
&i \mapsto \tau(i) \\
&s.t. \\
&\forall p \in P, \tau(p) \in A \\
&\forall a \in A, \tau(a) \in P \\
&\tau^{-1}(\tau(i)) = i
\end{aligned}$$

   With an abuse of notation I will sometimes use $\tau(p)$ for $t(\tau(p))$ to denote the type of the match.

2. Each Principal p designs a mechanism, in the form of a generic deterministic mechanism M_p, the action set for mechanism p is $\mathcal{A}_p$. The set of all mechanisms is called $\mathcal{M}_p$. At this stage every principal only knows CMN.

3. Each agent a chooses how to play from the mechanism $M_{\tau(a)}$ they face under the realized match. Each agent observes the menu, and since he knows the function $k(\cdot)$, he knows the technology of his match $k(\tau(a))$.

   A strategy of an agent a is a function of all payoff relevant information mapping to choice in mechanisms

$$C_a : \Pi_{p \in P} \mathcal{M}_p \rightarrow \Pi_{p \in P} \mathcal{A}^p$$
$$C_a : M_p \mapsto C_a \in \mathcal{A}_p$$

   The strategy profile of all agents are $C_A = \{C_a\}_{a \in A}$ which can be written as a function $C_A(M_P, \tau)$ of all menus offered and of the realized matching

4. Principals now observe types, mechanisms, and how each agent played. *Before* uncertainty is realized, they trade their claims to returns on an asset market, where the riskless asset L is available in zero supply.

This table summarizes, the choices each individual faces at a given time, and the information available to them.

When	**Who**	**What**	**Knowing What**
0	$p \in P$	M_p	CMN
1	$a \in A$	$C_a \in \mathcal{A}_{\tau(a)}$	$\text{CMN}, t(a), \tau(a), C_{\tau(a)}$
2	$p \in P$	θ_p	CMN, τ

Table 1: Timing

5.3 Payoffs

Let $\mathcal{T}$ be the set of possible matchings and τ its generic element.

Let θ be the portfolio held by an agent. Let $\theta = (\theta_R | \theta_L)$ Where θ_R is a P-dimensional vector of positive holdings of the P risky assets, whereas θ_L is the position an investor holds in the riskless asset.

The ex-ante utility from a portfolio θ fixing the matching τ and the contracts C_A, is given by

$$U_p^3(\theta, C_A, \tau) = \theta \cdot (\vec{\mu}(\tau, C_A)|1) - \frac{a}{2}\left(\theta_R' \mathbf{\Omega}(\tau, C_A)\theta_R\right)$$

Demand θ will depend on available assets and their prices (but prices are also a function of contracts).

Agents payoffs depend on the action chosen (and therefore on the type k of the principal and on their type t, $U_a = U_a^2(c_a, k, t)$:

$$U_a^2 = U_a^2(c_a, k, t)$$

Let F be the probability distribution induced by ν on the set of all possible matches $\mathcal{T}$. A Principal's *expected* utility when mechanisms are $(M_p)_{p \in P}$, actions are are C_A and he holds portfolio θ is

$$U_p^1(M_p, C_A, \theta) = \int_{\mathcal{T}} U_p^3(\theta, C_A, \tau) dF(\tau)$$

6 Equilibrium

6.1 Description

Because individuals take their decisions at each stage looking at the final payoffs, Equilibrium is more easily described starting from the final stage of the game.

Asset Market At this stage the realized matching is observed. Principals hold one unit of a security equal to their share of returns in their firm, and they all have the same information. The solution concept used here is that of Arrow-Debreu Equilibrium, in which quantity has an analytical expression, thanks to the assumptions on preferences and the presence of a riskless asset. The equilibrium portfolio and prices will be based on payoffs C_A and on the matching τ so they will be a function $(\theta, q)(C_A, \tau)$,

Contracting, the agents' turn Each agent a observes the mechanism offered to him, $M_{\tau(a)}$, and he knows his own type and the technology of the principal. This is all the payoff relevant information, so every agent is facing a choice between lotteries, and he is not playing against other players. They simply pick an action maximizing $U_a^2(\cdot)$. As noted their strategies will be functions $C_a(M_{\tau(a)}, k(\tau(a)), t(a))$.

Contracting, the principals' turn Each principal offers designs a mechanism, without knowing what agent they are going to be matched with. However they correctly forecast the strategy of each agent, and the outcome of asset markets, given menus and contracts. In other words they can forecast the equilibrium path for all profiles of mechanisms, and the specific outcome for all possible matchings. Principals at this stage play a game against each others, before the matching τ is realized. A mixed strategy is a lottery on possible mechanisms, $\tilde{\mathcal{M}} \in \Delta(\mathcal{M})$.

The flow of decisions is described schematically below, and the information available at each stage is summarized by the argument of the strategies.

$$\begin{matrix} \tilde{M}_P \\ \\ \tau \end{matrix} \rightarrow \tilde{C}_A(M_P, \tau)) \rightarrow \begin{matrix} C_a \\ \\ \tau \end{matrix} \rightarrow [(\theta, q)](C_A, \tau)$$

Based on this timeline we can write the utility in the first stage in this form:

$$V_p(\tilde{M}_p) = E_{\tilde{M}_p \times \tilde{C}_A(\cdot)} \left[U_p^1(\tilde{M}_p, \tilde{C}_A(\tilde{M}_p, \tilde{\tau}), \tilde{\tau}, \theta\{\tilde{C}_A(\tilde{M}_p, \tilde{\tau}), \tau\}) \right]$$

Note that $\tilde{\tau}$ is a random variable and remember that U_p^1 defined above includes an expectation with respect to $\tilde{\tau}$'s distribution.

6.2 Definition

An Equilibrium consists of

- A trading strategy θ^* for each Principal p and prices $q^* \in \mathbb{R}^{|P|}$ such that $[\theta^*, q^*](C_A, \tau)$ is an Arrow-Debreu Equilibrium for the symmetric information asset market taking place after contracting. Each principal is endowed with one unit of one asset so that the endowment of principal p is $w_p = [0, 0, ..., 1, ..., 0, 0]$ with 1 being in the pth position.

$$\begin{aligned} &\theta_p^*(C_A^*, \tau) \in arg \max_{\theta_p \in \mathbb{R}_+^P} U_p^3(\theta, C_A^*, \tau) \\ &s.t. \\ &q^*(C_A^*, \tau) \cdot \theta_p(C_A^*, \tau) \leq q^*(C_A^*, \tau) \cdot w_p \\ &\sum_{p \in P} \theta_p^* = [\mathbf{1}_P | 0] \end{aligned}$$

- For each agent a a strategy $C_a(M_{t(a)}, t, k)$ such that

$$C_a^*(M_p, \tau(a)) \in arg \max_{c \in M_p} U_a^2(c, \tau(a), t(a))$$

- For each principal p,a lottery $\tilde{M}_p^*$ of deterministic mechanisms such that

$$\begin{aligned}\text{supp}\left[\tilde{M}_p^*\right] &\subseteq arg \max_{M_p \in \mathcal{C}_p} V_p^*(M_p, M_{-p}^*) \\ &= U_p^1(M_p, M_{-p}^*, C_A^*(M_p, M_{-p}^*, \tilde{\tau}), \tilde{\tau}, \theta^*\{C_A^*(M_p, M_{-p}^*, \tilde{\tau}), \tilde{\tau}\})\end{aligned}$$

To prove existence of equilibrium I need the following result, a customized version of the mechanism design classic.

7 Revelation Principle and the Contract Space

7.1 MV Contracts

A binary Random Variable assuming values x_1, x_2, with $x_2 > x_1$ with probabilities $(\pi, 1-\pi)$ can be rewritten in terms of its mean and standard deviation:

$$x_1 = \mu - \sqrt{\frac{1-\pi}{\pi}}\sigma$$
$$x_2 = \mu + \sqrt{\frac{\pi}{1-\pi}}\sigma$$

Since in this paper, I focus on binary random variables, linear sharing rules are exhaustive. For natural economic reasons, I am interested only on sharing rules which are weakly monotone in outcomes. They can be expressed as $(\alpha, \beta) \in \mathbb{R} \times [0,1]$:

- P's share is $y = -\alpha + (1-\beta)x$

  This is a random variable paying taking the values

  $$\{-\alpha + (1-\beta)\mu - \sqrt{\frac{1-\pi}{\pi}}(1-\beta)\sigma, -\alpha + (1-\beta)\mu + \sqrt{\frac{\pi}{1-\pi}}(1-\beta)\sigma\}$$

- A's share is $y = \alpha + \beta x$ A random variable taking values

  $$\{\alpha + \beta\mu - \sqrt{\frac{1-\pi}{\pi}}\beta\sigma, \alpha + \beta\mu + \sqrt{\frac{\pi}{1-\pi}}\beta\sigma\}$$

In terms of Mean and Variance

- P's share is distributed with mean and variance $(-\alpha + (1-\beta)\mu, (1-\beta)^2\sigma^2)$
- A's share is distributed with mean and variance $(\alpha + \beta\mu, \beta^2\sigma^2)$

7.2 Result

Let an instance of the previously described economy be G, and the set of its equilibria $E(G)$. Consider now an economy G^T which is identical in all respects, but where principals are restricted to offer to agents menus of contracts of size $|T|$, instead of designing general game forms. I am going to show that $\mathcal{E}(G) = \mathcal{E}(G^T)$. In this way the strategy space of the Principals at the first stage will be finite dimensional.[4] The space of principals' strategies is now going to be $\mathcal{C}_p^{|T|}$.

Theorem 1. *The unrestricted menus economy G and the restricted menu economy G^T, have the same equilibria: $\mathcal{E}(G) = \mathcal{E}(G^T)$*

Proof. Let the contracts from $|T|$ sized menus be identified by $y_p(t), y_a(t)$, the shares of principal and agent when the agent picks contract t.

$\mathcal{E}(G) \subseteq \mathcal{E}(G^T)$ Consider an equilibrium $e = (\tilde{M}_P, C_A, \beta, \theta, q)$ now construct a strategy profile e^T and show it is an equilibrium for G^T: Let the menus M_p^T of $|T|$ contracts offered in $e(G^T)$ be made of contracts $\{y(t)\}_{t\in T}$:

$$y(t) = C_a(M_p, t, k(p))$$

For agents, it is an optimal to choose the contract $y(t)$ with their type t. Suppose not. Than it would be the case that for some t'

$$U(y(t')) > U(y(t))$$

By construction $C_a(M_p, t', k(p))$ played in M_p yields the same payoff as $y(t') = c$, which contradicts C_a being an equilibrium strategy.

The above described menus $\tilde{M}_p^T$ are optimal for every principal. Suppose not, then for some principal p there is a $|T|$ sized menu $\tilde{M}'$ such that

$$V_p^*(\tilde{M}'|\tilde{M}_{-p}^T) > V_p^*(\tilde{M}_P^T) = V_p^*(\tilde{M}_P)$$

The equality follows by construction, and since p could have offered $\tilde{M}'$ in the unrestricted economy, e could not be an equilibrium, which is a contradiction.

$\mathcal{E}(G^T) \subseteq \mathcal{E}(G)$

Consider an equilibrium e^T: the players strategies C_a^T are restrictions on of optimal equilibrium strategies in the unrestricted game. It is also that $\tilde{M}_P^T$, are equilibrium strategies also in the unrestricted game. Suppose it was not the case, then for some p there is an unrestricted mechanisms lottery $\tilde{M}$ such that

$$V_p^*(\tilde{M}|\tilde{M}_{-p}^T) > V_p^*(\tilde{M}_P^T)$$

[4] In this model, T is finite.

Note that $V_p^*(\tilde{M}|\tilde{M}_{-p}^T)$ can be attained by p by offering a lotteries of restricted menus $\tilde{M}'^T$ made of the following contracts y':

$$y'(t) = C_a(M, t, k(p))$$

This implies

$$V_p^*(\tilde{M}'|\tilde{M}_{-p}^T) > V_p^*(\tilde{M}_P^T)$$

which contradicts $\tilde{M}_P^T$ being an equilibrium for G^T □

To restrict the space of contracts to a set on which principals' payoffs will be continuous in equilibrium, it has to be that all and only the equilibria of the original unrestricted game are obtained in a game where Principals' are allowed to offer only an incentive compatible menu of T contracts. I will call this economy, G^{IC}.

Corollary 2. *The restricted menus economy G and the IC economy G^T, have the same equilibria $\mathcal{E}(G^T) = \mathcal{E}(G^{IC})$*

Proof. $\mathcal{E}(G^T) \subseteq \mathcal{E}(G^{IC})$ The contracts in the direct revelation menus in the proof of Theorem 1 are Incentive Compatible by construction.

$\mathcal{E}(G^{IC}) \subseteq \mathcal{E}(G^T)$ This part of the proof goes as the second part of the proof in Theorem 1: any outcome that a principal can achieve by menus of size T can be achieved by incentive compatibles menu of size T. □

8 Existence of Equilibrium

8.1 Monotonic Preferences

It is well known that Mean-Variance Preferences are not monotonic. This of course implies problems for the existence of equilibrium. In a standard CAPM setting, monotonicity of preferences is solved by imposing a bound on the variance aversion of every individual. Because I reduced the set of relevant mechanisms to linear contracts, it is possible to show that, if preferences are monotonic for given returns, they will be monotonic for any contracts in that economy.

Definition 1. *Let X be a generic Random Variable on the state space $S = (s_1, ..., s_n)$ taking values $(x_1, ..., x_n)$. We say that $U(X)$ is monotonic if $\frac{\partial U}{\partial x_i} > 0, \forall i$.*

Lemma 3. *Consider the preferences induced by the utility function*

$$U(X) = E(X) - \frac{a}{2} Var(X)$$

They are monotonic on a set of variables $\mathcal{X}$ defined on a finite state space S, if

$$a < \min_{X,s} \frac{1}{|x_s - \mu_X|}$$

Proof. The proof amounts to checking (by differentiating) under which conditions on a the utility function is increasing. □

Lemma 4. *If preferences are monotonic for all feasible portfolios in an economy with assets characterized by returns (μ, Ω), then they will be monotonic for any contracts $(\alpha_p, \beta_p)_{p\in P}$.*

Proof. For preferences to be monotonic for all feasible portfolios it has to be that

$$\begin{aligned} a <& \frac{1}{|\max \sum_{p\in P} \theta_p x_p - \sum_{p\in P} \theta_p \mu_p|} \\ =& \frac{1}{|\max \sum_{p\in P} \theta_p (x_p - \mu_p)|}. \end{aligned}$$

Where the max is taken across portfolios θ such that $\theta_p \in (0, 1)$ and outcomes $x_p \in supp(X_p)$. Note that

$$\begin{aligned} & \sum_{p\in P} \theta_p [(-\alpha_p + (1 - \beta_p)] x_p - [-\alpha_p + (1 - \beta_p) \mu_p)] \\ =& \sum_{p\in P} \theta_p ((1 - \beta_p) x_p - (1 - \beta_p) \mu_p) \\ =& \sum_{p\in P} \theta_p (1 - \beta_p) (x_p - \mu_p) \end{aligned}$$

I claim that

$$\max |\sum_{p\in P} \theta_p (1 - \beta_p) (x_p - \mu_p)| < \max |\sum_{p\in P} \theta_p (x_p - \mu_p)|$$

Note that the solution to the maximization on both sides is going to be reached at the aggregate market portfolio so that the previous is equivalent to

$$\max |\sum_{p\in P} (1 - \beta_p) (x_p - \mu_p)| < \max |\sum_{p\in P} (x_p - \mu_p)|$$

Since it will also be the case that at the maximum all the x_p's chosen will be greater (or smaller) than the μ_p's so that

$$\max \sum_{p \in P} (1 - \beta_p) \,|\, (x_p - \mu_p) \,| < \max \sum_{p \in P} |\, (x_p - \mu_p) \,|$$

Observing that $\beta_p \in (0, 1)$ concludes the proof □

8.2 The existence result

Theorem 5. *If the mean-variance preferences are monotonic for an asset market economy characterized by the mean vector μ and variance-covariance matrix Ω then there exists an equilibrium in the CAPM contracting economy.*

Proof. I am going to use a well known fixed point result by Glicksberg (1952) to show that there is an equilibrium in the first stage of the game, given that the asset market develops as predicted by the CAPM model.

I need to show that

1. The strategy space $\Delta(\mathcal{M}^{MV})$ is a convex, compact subset of a locally convex Hausdorff space.
2. The best response correspondence of all principals is upper hemi-continuous, convex valued, and nonempty.

For the first part note that the space of Incentive Compatible menus $\mathcal{M}^{MV}$ is a subset of a Euclidean space. It is closed because it is defined by a finite number of weak inequalities, and it is bounded because the larger set of feasible contracts are bounded. Hence it is compact.

The space of lotteries (identified with Borel probability measures) over these Menus is of course convex. It is also compact with respect to the weak* topology. This space of probabilities is a subset of the space of continuous functions $\mathcal{C}(\mathcal{M}^{MV})$, which is locally convex (and Hausdorff) with respect to the weak* topology.[5]

For the second part, convexity of the best response correspondence follows because the returns under two different matchings are independent: if $\mu^*(\tilde{M}_P), \sigma^{*2}(\tilde{M}_P)$ are the mean and variance of returns of the portfolio holdings induced by a menu profile, suppose $\tilde{M}$ and $\tilde{M}'$ are both maximizers

$$\mu^*(\tilde{M}, \tilde{M}_{-p}) - \frac{a}{2}\sigma^{*2}(\tilde{M}, \tilde{M}_{-p})) = \mu^*(\tilde{M}', \tilde{M}_{-p}) - \frac{a}{2}\sigma^{*2}(\tilde{M}', \tilde{M}_{-p}))$$

[5]For a treatment of these and other results on the weak topologies, and also to see the theorems of Berge and Glicksberg, see Aliprantis, Border (2005)

If p plays $\delta\tilde{M}_p+(1-\delta)\tilde{M}'_p$, the mean of the outcome will be $\delta\mu^*(\tilde{M},\tilde{M}_{-p})+(1-\delta)\mu^*(\tilde{M}',\tilde{M}_{-p})$ and the variance $\delta\sigma^*(\tilde{M},\tilde{M}_{-p})+(1-\delta)\sigma^*(\tilde{M}',\tilde{M}_{-p})$. Principal p's utility will be

$$\delta\mu^*(\tilde{M},\tilde{M}_{-p})) + (1-\delta)\mu^*(\tilde{M}',\tilde{M}_{-p})) - \frac{a}{2}\left[\delta\sigma^{*2}(\tilde{M},\tilde{M}_{-p})) + (1-\delta)\sigma^{*2}(\tilde{M}',\tilde{M}_{-p}))\right] =$$
$$\delta\left[\mu^*(\tilde{M},\tilde{M}_{-p})) - \frac{a}{2}\sigma^{*2}(\tilde{M},\tilde{M}_{-p}))\right] + (1-\delta)\left[\mu^*(\tilde{M}',\tilde{M}_{-p})) - \frac{a}{2}\sigma^{*2}(\tilde{M}',\tilde{M}_{-p}))\right] =$$
$$\mu^*(\tilde{M},\tilde{M}_{-p})) - \frac{a}{2}\sigma^{*2}(\tilde{M},\tilde{M}_{-p}))$$

which implies that any convex combination of two maximizers is also a maximizer, so that the best response correspondence is convex. I will use Berge's Maximum theorem show that it is also non empty, compact-valued and upper hemi-continous.

To apply the maximum theorem to individuals' best response, it has to be that constraints vary continuously with other principals' strategies, and that the payoff function is continuous in one's own actions.

First note how the constraints correspondence is constant with respect to other principals strategies, and is therefore continuous. Also note how the constraints correspondence maps to the space of Borel probability measures on menus, which is a Hausdorff space as noted above.

By Lemma 4, if the preferences are monotonic for (μ, Ω), they are going to be monotonic for the asset markets resulting from all possible contracts C_P. When the CAPM equilibrium exists, the indirect utility from a contract profile in the CAPM function is continuous. As noted in Nielsen (1990). Nielsen shows that an equilibrium with *some* positive prices exists. The utility from the market $q_p - \frac{\sum_{p\in P} q_p}{P} + \frac{1}{|P|}\left(\sum_{p\in P}\mu_p\right) - \frac{a_p}{2P^2}\mathbf{1'\Omega 1}$ could be discontinuous only when *all* prices q_p are zero, making the denominator in all individual shares undefined, but this is exactly ruled out.[6]

Taking expectation with respect to the probability of matches over these indirect utilities yields a continuous functional on the domain of lotteries on IC and IR menus.

By the maximum theorem the best response correspondence of each player is now UHC and compact valued, which implies that the game best response is as well.

By Glicksberg's theorem there is a fix point, which is an equilibrium. □

[6] In the literature briefly reviewed by Nielsen (1990), one can find many sufficient conditions for the existence of CAPM equilibrium, most of them deal with the possibility of satiation of preferences. Things are particularly simple when returns are bounded (which includes this model): monotonicity and local non satiation are guaranteed by a low enough risk aversion

9 The Insurance Effect of Markets

In this section I give sufficient conditions for two-types to exhibit the insurance effect of the example. I restrict attention at the two-type case.

Lemma 6. *Consider an economy where returns are uncorrelated ($\Omega(\tau)$ is diagonal) and principals have the same risk aversion (a_p is the same for all ps) induces a lower risk aversion. If the principals are allowed to trade their claims, they will act as if their utility function were*

$$E_t\left[-\alpha_t + (1-\beta_t)\,\mu_t - \frac{a}{2}\frac{2N-1}{N^2}\,(1-\beta_t)^2\,\sigma_t^2\right]$$

Proof. Consider a given matching, the utility obtained by a principal, say is

$$\begin{aligned}
&U(\alpha_p,\beta_p) = \\
&\alpha_p + \beta_p\mu_p - \frac{a}{P}\beta_p^2\sigma_p^2 - \frac{\sum_{k\in P}\alpha_k + \beta_k\mu_k}{P} + \frac{a}{P^2}\sum_{k\in P}\beta_k^2\sigma_k^2 + \\
&+ \frac{\sum_{k\in P}\alpha_k + \beta_k\mu_k}{P} - \frac{a}{2P^2}\sum_{k\in P}\beta_k^2\sigma_k^2 = \\
&\alpha_p + \beta_p\mu_p - \frac{a}{P}\beta_p^2\sigma_p^2 + \frac{a}{2P^2}\sum_{k\in P}\beta_k^2\sigma_k^2
\end{aligned}$$

So for the point of view of the optimization problem the utility function can be trimmed down to

$$-\alpha_p + (1-\beta_p)\,\mu_p - \frac{a}{2}\frac{2N-1}{N^2}\,(1-\beta_p)^2\,\sigma_p^2$$

Taking expectation over possible matchings yields the desired functional form. □

Theorem 7. *When information is complete and symmetric, the variance of optimal contracts is decreasing in the size of the markets N for both agents.*

$$\frac{\partial\beta_i^*(N)}{\partial N} < 0, \forall i$$

Proof. The principal problem is given by

$$\begin{aligned}
&\max_{(\alpha_t,\beta_t)_{t=1}^T}\sum_{t=1}^T q_t\left[-\alpha_t + (1-\beta_t)\mu_t - \frac{a}{2}\frac{2N-1}{N^2}(1-\beta_t)^2\sigma_t^2\right] \\
&\quad s.t.\ \ IR_t : \alpha_t + \beta_t\mu_t - \frac{a}{2}\beta_t^2\sigma_t^2 \geq \overline{u}
\end{aligned}$$

Substituting the constraints in the objective function we obtain

$$\max_{(\beta_t)_{t=1}^T} \sum_{t=1}^T q_t \left[\beta_t \mu_t - \frac{a}{2} \beta_t^2 \sigma_t^2 - \overline{u} + (1-\beta_t)\mu_t - \frac{a}{2} \frac{2N-1}{N^2} (1-\beta_t)^2 \sigma_t^2 \right]$$

Differentiating with respect to each β_t we obtain

$$\beta_t^* = \frac{2N-1}{N^2+2N-1}, \forall t$$

whose derivative is

$$\frac{\partial \beta_t^*}{\partial N} = \frac{1-2N}{\left(N^2+2N-1\right)^2} < 0, \forall N > 1$$

which proves the claim □

Definition 2. *Types* $((\mu_1, \sigma_1), (\mu_2, \sigma_2))$ *satisfy* Increasing Differences *if*

$$U_1(\alpha, \beta) - U_1(\alpha', \beta') > U_2(\alpha, \beta) - U_2(\alpha', \beta'), \forall \beta > \beta'$$

When agents have mean variance preferences with the same risk tolerance this amounts to

$$\mu_1 - \mu_2 - a\left(\sigma_1^2 - \sigma_2^2\right) \geq 0$$

Proposition 8. *The following cases imply ID:*

- *Agents have the same mean and different variance*
- *Agents have different mean and the same variance*
- *Both agents generate the same outcomes, but the probabilities of success are different.*

The proof amounts to verifying the definition.

Theorem 9. *If Increasing Differences is satisfied, the variance of optimal contracts is decreasing in the size of the markets N for all agents.*

$$\frac{\partial \beta_i^*(N)}{\partial N} < 0, \forall i$$

Proof. The maximization problem of the principal is

$$\max_{((\alpha_1,\beta_1),(\alpha_2,\beta_2))} q\left[-\alpha_1+(1-\beta_1)\mu_1-\frac{a}{2}\frac{2N-1}{N^2}(1-\beta_1)^2\sigma_1^2\right]+$$
$$(1-q)\left[-\alpha_2+(1-\beta_2)\mu_2-\frac{a}{2}\frac{2N-1}{N^2}(1-\beta_2)^2\sigma_2^2\right]$$
$$s.t.\quad IR_1:\alpha_1+\beta_1\mu_1-\frac{a}{2}\beta_1^2\sigma_1^2\geq\overline{u}$$
$$IR_2:\alpha_2+\beta_2\mu_2-\frac{a}{2}\beta_2^2\sigma_2^2\geq\overline{u}$$
$$IC_1:\alpha_1+\beta_1\mu_1-\frac{a}{2}\beta_1^2\sigma_1^2\geq\alpha_2+\beta_2\mu_1-\frac{a}{2}\beta_2^2\sigma_1^2$$
$$IC_2:\alpha_2+\beta_2\mu_2-\frac{a}{2}\beta_2^2\sigma_2^2\geq\alpha_1+\beta_1\mu_2-\frac{a}{2}\beta_1^2\sigma_2^2$$
$$Stock:\beta_i\in(0,1)$$

By Increasing Differences, we can infer by standard arguments[7], that IC_2 won't be binding. and IR_2 can be substituted by a monotonicity constraint. $\beta_1\geq\beta_2$. We also know that β_1 is going to coincide with the first best solution (optimal risk sharing), so that it will satisfy the *Stock* constraint. Always by Increasing Differences we know that IC_1 and IR_1 will be binding.

$$\max_{((\alpha_1,\beta_1),(\alpha_2,\beta_2))} q\left[-\alpha_1+(1-\beta_1)\mu_1-\frac{a}{2}\frac{2N-1}{N^2}(1-\beta_1)^2\sigma_1^2\right]+$$
$$(1-q)\left[-\alpha_2+(1-\beta_2)\mu_2-\frac{a}{2}\frac{2N-1}{N^2}(1-\beta_2)^2\sigma_2^2\right]$$
$$s.t.\quad IR_1:\alpha_1+\beta_1\mu_1-\frac{a}{2}\beta_1^2\sigma_1^2=\overline{u}$$
$$IC_1:\alpha_1+\beta_1\mu_1-\frac{a}{2}\beta_1^2\sigma_1^2=\alpha_2+\beta_2\mu_1-\frac{a}{2}\beta_2^2\sigma_1^2$$
$$Stock+Monotonicity:\beta_2\in[0,\beta_1]$$

First solve the problem for the first two constraints and then check for the last one to be satisfied.

The first order conditions for the type 1 are the same as in the complete information problem, yielding

$$\beta_1=\frac{qa\frac{2N-1}{N^2}\sigma_1^2}{qa\left(\frac{2N-1}{N^2}\sigma_t^2+\sigma_t^2\right)}=\frac{2N-1}{N^2+2N-1}$$

hence we have that β_1 is a decreasing function of N.

[7] See Bolton-Dewatripont (2005), chapter 2

For type 2

$$\beta_2 = \frac{q(\mu_2-\mu_1)+(1-q)a\frac{2N-1}{N^2}\sigma_2^2}{aq\left(\sigma_2^2-\sigma_1^2\right)+(1-q)a\left(\frac{2N-1}{N^2}\sigma_2^2+\sigma_2^2\right)}$$
$$= -\frac{(1-q)a\frac{2N-1}{N^2}\sigma_2^2-q(\mu_2-\mu_2)}{(1-q)a\frac{N^2+2N-1}{N^2}\sigma_2^2-aq\left(\sigma_1^2-\sigma_2^2\right)}$$

If $\beta_1 \geq \beta_2$, this is the solution.

It might be the case that $\beta_2 < 0$ or $\beta_1 < \beta_2$ (the third constraint is binding).

As it is usually the case, when $\beta_2 < 0$, β_2^* will be equal to zero at the optimum and β_1^* is going to coincide with the solution to the complete information problem.

A little more attention is needed to reach the same conclusion for the case in which $\beta_1 < \beta_2$. This implies that IR_1 is violated, so we know that it is going to be binding at the optimum. Consider IR_1, IC_1, IR_2 as a system of equalities.

$$IR_1 : \alpha_1 + \beta_1\mu_1 - \frac{a}{2}\beta_1^2\sigma_1^2 = \overline{u}$$
$$IR_2 : \alpha_2 + \beta_2\mu_2 - \frac{a}{2}\beta_2^2\sigma_2^2 = \overline{u}$$
$$IC_1 : \alpha_1 + \beta_1\mu_1 - \frac{a}{2}\beta_1^2\sigma_1^2 = \alpha_2 + \beta_2\mu_1 - \frac{a}{2}\beta_2^2\sigma_1^2$$

Solving for β_2 gives

$$\beta_2(\mu_1-\mu_2) - \frac{a}{2}\beta_2^2\left(\sigma_1^2-\sigma_2^2\right) = 0$$

This equation has two solutions:

$$\beta_2 = \left(0, \frac{2(\mu_1-\mu_2)}{a\left(\sigma_1^2-\sigma_2^2\right)}\right)$$

By ID the non zero solution is at least 2, which is ruled out as β_2^* has to be smaller than 1. Hence $\beta_2^* = 0$ in this case too.

With the whole picture in mind, two things are left to show to conclude that β_2^* is a decreasing function of N.

1. That the interior solution is decreasing in N
2. That the only possible change as N increases is from an interior solution to the corner solution and never vice versa.

To prove claim 1, it is sufficient to differentiate and rearrange the expression for the interior solution.

$$\frac{\partial \beta_2}{\partial N} = \frac{(1-q)\,a\frac{2N-2N^2}{N^4}\sigma_2^2 q\left(\sigma_2^2-\sigma_1^2\right)+(1-q)\,a\left(\frac{2N-1}{N^2}+1\right)\sigma_2^2 -}{\left(aq\left(\sigma_2^2-\sigma_1^2\right)+(1-q)\,a\left(\frac{2N-1}{N^2}\sigma_2^2+\sigma_2^2\right)\right)^2}+$$
$$-\frac{q_t a\frac{2N-2N^2}{N^4}\sigma_t^2\left[q\left(\mu_t-\mu_1\right)+(1-q)\,ta\frac{2N-1}{N^2}\sigma_t^2\right]}{\left(aq\left(\sigma_t^2-\sigma_1^2\right)+(1-q)\,ta\left(\frac{2N-1}{N^2}\sigma_t^2+\sigma_t^2\right)\right)^2}$$

The sign of the numerator is going to be the opposite of on the sign of the following expression

$$q\left[(\mu_1-\mu_2)-a\left(\sigma_1^2-\sigma_2^2\right)\right]+(1-q)a\sigma_2^2$$

This is always positive when Increasing Differences holds (the term in square brackets is positive). Hence the interior solution candidate is decreasing in N.

Now to claim 2. Bearing in mind that β_1^* always coincides with the first best solution. We need to study the sign of the expression $\beta_1-\beta_2$, and show that as N increases it can go from positive to negative, but not the other way.

$$\beta_1-\beta_2 = \frac{2N-1}{N^2+2N-1}-\frac{(1-q)a\frac{2N-1}{N^2}\sigma_2^2-q\left(\mu_1-\mu_2\right)}{(1-q)a\frac{N^2+2N-1}{N^2}\sigma_2^2-aq\left(\sigma_1^2-\sigma_2^2\right)}=$$
$$\frac{(2N-1)\,q\left[(\mu_1-\mu_2)-a\left(\sigma_1^2-\sigma_2^2\right)\right]+N^2q\left(\mu_1-\mu_2\right)}{(N^2+2N-1)\left[(1-q)\,a\left(\frac{N^2+2N-1}{N^2}\right)\sigma_2^2-aq\left(\sigma_1^2-\sigma_2^2\right)\right]}$$

We need to consider two cases

$\mu_1<\mu_2$. ID implies that $\sigma_1<\sigma_2$ so that the denominator will always be positive. The sign is determined by the numerator. The numerator is a decreasing function of N since the coefficient of N^2,$(\mu_1-\mu_2)$ is negative. Hence the sign can only change from positive to negative.

$\mu_1\geq\mu_2$ implies that the second term of the numerator is positive ID implies that the second term is positive. Hence the sign depends on the denominator. The sign of the denominator is the sign of the expression

$$(1-q)a\left(\frac{2N-1}{N^2}+1\right)\sigma_2^2-aq\left(\sigma_1^1-\sigma_2^2\right)$$

This is a decreasing function of N. Hence the sign can only change from positive to negative.

So we have shown that β_2^* will be a non-decreasing function of N □

9.1 Effects on Welfare and Asset Markets

As it is usually the case, asymmetric information entails a loss of efficiency.

Corollary 10. *Under the assumptions of Theorem 9, the equilibrium is inefficient compared to the complete information case.*

Proof. The non-transferable utility part of the contract, doesnt coincide with the first best, hence without the IC constraints, Pareto-improving transfers would be possible. □

From the point of view of asset markets this inefficiency takes the form of a higher market risk.

Corollary 11. *In equilibrium every firms issues securities that are (weakly) riskier than optimal riskier. Hence the aggregate risk is also higher than optimal.*

Proof. In the proof of Theorem 9 it was shown that the β_1^* will be the same as in the first best optimum and β_2^* is always lower. As a result $(1-\beta_2^*)^2$ will be higher. Because the previous statement is true for every principal the aggregate risk $\sum_{p \in P} \left(1-\beta_p^*\right)^2 \sigma_p^2$ will also be higher. □

Corollary 12. *For any principal-agent pair, there is a number $\overline{N}$), such that any market with $N \geq \overline{N}$ traders induces more efficient contracts.*

Proof. The claim is trivially satisfied for β_1^*, since it coincides with the first best optimum. For β_2^* consider $\Delta_2(N) = \beta_1^*(N) - \beta_2^*(N)$. Since $\beta_1^*(N)$ goes to zero as N gets large, there must be a $\overline{N}$ such that $\beta_1^*(\overline{N}) < \Delta_2(1)$ for all $N > \overline{N}$. Because $\Delta_2(N)$ is a positive quantity smaller than $\beta_1 * (N)$ for all N, it is also the case that this implies $\Delta_2(\overline{N}) < \Delta_2(1)$, which proves the claim. □

Note how this last statement needs to be a limiting statement to take into account the possibility that $\beta_2^*(N)$ "jumps" to zero at some N and Δ_2 would not necessarily be decreasing at that discontinuity.

10 Conclusion

This paper integrates a model of principal-agent interaction with asset markets. Principals and Agents are randomly matched. Each pair produces random returns, whose distribution is known only to the agent at the contracting stage. Every Principal offers a menu of contracts to the Agent he is matched with, and the Agents make their pick. What marks the difference from the standard contracting model is that Principals have access to an asset market on which they trade their shares of returns.

I present a general framework and define a notion of equilibrium. I prove existence a revelation principle result, which I use to prove existence of equilibrium. Under standard assumptions of contract theory, I study the interactions of financial markets on contracts. The existence of markets, induces less risky compensation for agents. Theorem 9 also implies that as markets get larger and diversification opportunities multiply, contracts become less and less risky.

Contracting inside firms induces excessive aggregate risk in an economy, however the size of this inefficiency is reduced by a large enough market. As noted this is a limiting result and it leaves the open question of the behavior in small markets.

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