Incomplete Markets and the Yield Curve*

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October 29, 2007

Abstract

We present a general equilibrium model with incomplete markets and borrowing constraints to study the interest rate term structure. Agents face both an aggregate risk and an idiosyncratic risk of unemployment, which is not insurable. We derive a reduced heterogeneity equilibrium, which allows for analytical bond prices, whatever their maturities. The model exhibits a liquidity channel, through which the bond supply affects the shape of the yield curve: A larger bond volume raises both the level and the slope of the curve. We additionally derive the implications of credit constraints for the interest rate term structure and notably their contributions to the rejection of the expectation hypothesis. We finally rank our equilibria according to the bond volumes using a Pareto criterion. If a larger bond volume is welfare enhancing from an ex ante point of view, it may deter the utility of the poorest agents, if the discount factor is low.

Keywords: incomplete markets, yield curve, credit constraints.

JEL codes: E21, E43, G12.

1 Introduction

Mehra and Prescott (1985) have the first reported the limitation of the standard Lucas (1978) asset pricing model in matching both the equity premium and the risk free rate. A wide range of financial imperfections has been investigated to overcome Lucas’ model shortcomings and to

*Acknowledgments: We are indebted to Gabrielle Demange, François Gourio, Guy Laroque, Caroline Mueller, and Monika Piazzesi for helpful suggestions. We also thank participants at the PSE Lunch seminar, at the joint IHEC-INSEAD-PSE Workshop, at the 2007 Meeting of the Society for Economic Dynamics, and at the 2007 ESEM Conference for valuable comments.

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replicate the price of risk. The market incompleteness is one of the channels, which is likely to account for both puzzles. Huggett (1993) considers an endowment economy, where agents face an idiosyncratic risk of income and smooth their consumption using a risk-free asset. Moreover, agents are forbidden to borrow. In consequence, they must purchase the available asset to self-insure themselves against the income shock. This builds up a new motive for holding assets and turns up the asset demand, as well as the asset price. This precautionary saving due to the market incompleteness and credit constraints, contribute to explain the risk-free rate puzzle. Constantinides and Duffie (1996) considers also an endowment economy with income shocks, but without credit constraints. They prove that these idiosyncratic shocks, along with a particular process for dividends, help to reproduce simultaneously the equity premium and the risk-free rate. More recently, Gomes and Michaelides (2006) consider an economy with an idiosyncratic risk, borrowing constraints and aggregate shocks. They manage to simultaneously match not only the equity premium and the risk-free rate, but also individual asset holdings.

Nevertheless, if the risk premium has been successfully addressed in a large number of papers with the market incompleteness, the term structure of interest rates has received much less attention. The yield curve is one of the few direct sources of information about agents’ expectation about future economic activity. Unlike the equity premium, the risk embedded in the curve has two dimensions. The first dimension, which is common to the equity, is the time series dynamics of the risk. The second one is a cross-sectional aspect, which links relative prices of various maturities together. Initial models of the term structure, such as Cox, Ingersoll, and Ross (1985) assumed a frictionless complete market environment, which entails that the term structure of interest rates only changes with the realization of an aggregate technology risk. Other yields are computed using the no-arbitrage assumption. From then on, a vast literature has generalized no-arbitrage restrictions to derive the dynamics of the term structure from the short rate determined with a limited number of factors. These models present several attractive features, as analytical bond prices, and a good fit of the data. However, they do not provide any micro-foundation for the term structure nor any general equilibrium understanding. The term structure of interest rates has not been studied with market incompleteness and infinite horizon yet, because of specific difficulties. The equilibrium must be notably solved with a large number of maturities, to determine the shape of the yield curve. Analytical results are very difficult to obtain and even computational techniques are not yet available. To circumvent the difficulties with infinite horizon, three-period models have been developed, which allow

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To the best of our knowledge, since Krussel and Smith (1997), the computational literature, with incomplete market and aggregate shocks, deals only with two assets at most.
to identify new mechanisms generated by market incompleteness. Heaton and Lucas (1992) develop such a model and find that uninsurable income shocks contribute to increase the term premium and thus increase the slope of the yield curve. Holmström and Tirole (1997) also use a three-period model, but they assume that news about the state of the world occur between date 0 and date 1. These models underline that the term structure is affected by the volume of securities, which are available for self-insurance against the various shocks in the economy. To be able to use computational techniques, Seppälä (2004) study an endowment economy with two agents, complete market and endogenous borrowing constraints à la Alvarez and Jermann (2000). In this class of debt-constrained economy (Kehoe and Levine (2001)), liquidities do not play the role, identified in the three-period models.

In this paper, we prove that incomplete markets have important implications not only for explaining the equity premium, but also for understanding the interest rate term structure. The market incompleteness makes aggregate and idiosyncratic risks interact, which allows the volume of liquidities and the probability of bad idiosyncratic shocks to affect the shape of the yield curve. In this paper, we present an equilibrium model of the yield curve in a production economy with infinite horizon and incomplete markets. We are able to prove the existence of the equilibrium with an arbitrary high number of maturities. We also obtain analytical solutions and identify the new effects for the shape of the yield curve, which stem only from market incompleteness. In this model, agents face both uninsurable idiosyncratic and aggregate shocks. They cannot issue contingent debt securities on their future incomes, but they can purchase riskless bonds of various maturities. These bonds are issued by the government, which collects non distorting taxes. The model can be solved because of two assumptions, which were used separately in the literature. First, agents, who face a bad idiosyncratic shock, are supposed to liquidate their portfolio and not to decumulate progressively their wealth (Kehoe et al., (1992)). Second, the utility of private agent is assumed to be separable and linear in leisure (Scheinkman and Weiss, (1986)). Both assumptions, used simultaneously, reduce the heterogeneity to only four types of agents, whereas incomplete market model deals in general with an infinitely dimensional distribution of agents' types (Scheinkman and Weiss, (1986); Huggett, (1993) among others). This tractable equilibrium with limited heterogeneity is thus of independent interest in the incomplete market literature.

This analytical structure yields three main results. First, for a constant value of bonds, an increase in the idiosyncratic risk of having a low revenue and thus of facing credit constraints shifts the term structure downwards. Indeed, the demand for titles to self-insure is higher,
what decreases interest rates of all maturities. Second, for a given level of idiosyncratic risk, a larger title volume shifts the term structure upwards and puts the slope of the yield curve up. Larger volumes soften indeed the self-insurance motive. The demand for titles goes down, what raises interest rates. Moreover, although bonds are reimbursed at maturity with certainty, they bear the risk of being liquidated before maturity at an unwanted price, which depends on the aggregate shock. The larger the volumes, the riskier this liquidation. In consequence, the slope of the yield curve steepens with the bond volume. We also prove that volumes affect the variance of the interest rates at any maturity, because they affect differently the self-insurance motive according to the realization of the aggregate shock. Third, the departure from complete market assumption contributes to the rejection of the expectation hypothesis: The higher the probability of the uninsurable bad shock, the larger the bias of forward rates as predictors of future spot rates. This result summarizes the effects on the term structure of the departure from complete markets explained in the two other results.

Finally, in our model, the size of bond issuance can be arbitrarily chosen and government taxes adjust to balance bond repayments and issuances. Provided that the bond volumes are not too large, in the sense that poor agents remain credit constrained, there exists for each bond issuance policy an equilibrium with four agents. We prove that a larger bond volume enhances the welfare from an ex ante point of view, but it may deter the welfare of poorest agents, if the discount factor is low. Indeed, high volumes are useful only for the richest agents who have the incentive to self-insure.

We investigate also the empirical behavior of our model. First, we calibrate the model using macroeconomic data in order to assess the order of magnitudes of our findings. Second, we use a panel regression to measure to what extend data support our theoretical findings. We show notably that the influence of bond volumes and credit constraints on the level and the slope of the curve is in line with our predictions.

The rest of paper is organized as follows. The section 2 describes our economy. In the third section, we use the two technical assumptions to derive the four agents’ types equilibrium and we deduce the analytical bond pricing structure. We prove the equilibrium existence in section 4. We provide intuitions regarding implications of the model on the yield in a small model with two maturities in the fifth section. The section 6 generalizes preceding insights to the whole curve. We derive notably the implications of credit constraints and bond volumes on the shape curve and the rejection of the expectation hypothesis. Our last result consists in ranking our

\footnote{See Fleming (2002) for empirical evidence of supply on interest rates and Jegadeesh (2002) for a discussion with other empirical evidence.}
equilibria using a Pareto criterion. The section 7 finally concludes.

2 The Economy

The economy is populated by a unit mass of agents and firms, who interact in perfectly competitive markets. A government exists and issues bonds of arbitrarily high maturities. Those bond prices build up the yield curve, which we investigate deeper in the rest of the paper. The government also raises taxes in order to balance bond issuances and bond repayments, but it does not run any expenditure.

2.1 Uncertainty

We begin with describing the shocks, either idiosyncratic or aggregate, that the agents are likely to face.

2.1.1 Idiosyncratic risk

Each agent faces an idiosyncratic risk of unemployment. The status of an agent $i$ at date $t$ on the labor market is denoted $e_i^t$. This variable takes the value 1 ($e_i^t = 1$), if the agent is employed. On the opposite, it is equal to 0 ($e_i^t = 0$), when the agent is jobless. Each agent’s employment status follows an independent first-order Markov chain, with a $2 \times 2$ transition matrix $\Pi$:

$$
\Pi = \begin{bmatrix}
\alpha & 1 - \alpha \\
1 - \rho & \rho
\end{bmatrix}
$$

where $0 < \alpha, \beta < 1$

The interpretation of transition probabilities is straightforward. For example, the probability for an employed agent to be jobless in the next period is $1 - \alpha$.

The initial probability distribution is summarized in a row vector $\omega_0 = [\omega_0^e \, \omega_0^u]$. The quantity $\omega_0^e$ (resp. $\omega_0^u$) is the probability at date 0 that the agent $i$ is employed (resp. unemployed). It also interprets as the initial share of employed (resp. unemployed) agents in the economy. Due to the Markovian structure, the probability distribution at a given date $t$ expresses as $\omega_0 \Pi^t$. This distribution converges for $t \to \infty$ towards the stationary distribution $\omega = [\omega^e \, \omega^u]$, where:

$$
\omega^e = \frac{1 - \rho}{2 - \rho - \alpha} \quad \text{and} \quad \omega^u = \frac{1 - \alpha}{2 - \rho - \alpha}
$$

(1)

For sake of simplicity, we assume that the initial distribution is equal to the stationary one:
ω₀ = ω. Shares of employed and unemployed agents in the population always remain constant and respectively equal to ωₑ and ωᵤ. This assumption allows us to avoid considering transitory dynamics.

The agent i’s history of individual shocks up to date t is gathered in the vector eᵢ,t = \{eᵢ,0, .., eᵢ,t\} ∈ \{0, 1\}^t = Eᵢ, where Eᵢ is the set of all possible individual histories up to date t. We use the notation eᵢ,t+1 ⪰ eᵢ,t to indicate that eᵢ,t+1 is a possible continuation of eᵢ,t. The function µᵢ⁻ : Eᵢ → [0, 1], t = 0, 1, ... is the probability measure over individual histories for the agent i, which is consistent with the transition matrix Π and with the initial probability distribution ω. For example, µᵢ⁻(eᵢ,t) is the probability that the agent i experiences the history eᵢ,t at date t.

2.1.2 Aggregate risk

The economy faces an aggregate shock, whose value at date t is hₜ. This shock can either be high (hₜ = h) or low (hₜ = l). The history of aggregate shocks up to date t is denoted hᵗ, while Hᵗ is the set of all possible histories up to date t. The aggregate shock evolves according to a first-order Markov chain with the transition matrix T:

$$T = \begin{bmatrix} \pi^h & 1 - \pi^h \\ 1 - \pi^l & \pi^l \end{bmatrix}$$

(2)

Although it is not necessary, we make the following assumption to avoid uninteresting cases.

Assumption A \(\pi^h + \pi^l > 1\)

This assumption states that the economy does not alternate “too often” between good and bad states. In that sense, the aggregate shock is persistent³.

The invariant distribution associated to the transition matrix T is denoted Φ = [Φₗ Φₜ]. We assume that the initial probability distribution across both aggregate states is Φ. We denote νₜ the probability measure over histories up to date t, consistent with the transition matrix T, and the initial distribution Φ: νₜ : Hᵗ → [0, 1], t = 0, 1, ... Similarly to the idiosyncratic case, νₜ (hᵗ) is the probability that the history hᵗ occurs at date t, and hᵗ+1 ⪰ hᵗ indicates that hᵗ+1 is a possible continuation of hᵗ.

³This assumption appears as empirically relevant: Using US quarterly data on GNP, Hamilton ((1994), chap 22) finds that \(\pi^h + \pi^l = 1.65\).
2.2 Credit Market

2.2.1 Description

Agents have access to a credit market, where riskless zero-coupon bonds are the only available assets. Their maturities vary from 1 to \( n \geq 1 \), which can be arbitrarily high. A bond of maturity \( k > 1 \) at period \( t \), becomes a bond of maturity \( k - 1 \) at date \( t + 1 \) and it finally yields one unit of good at date \( t + k \). The price of this bond at date \( t \) is \( p_{t,k}(h^t) \), when the aggregate history is \( h^t \). To simplify the notations, we define the price of a bond of maturity 0 as its payoff: \( p_{t,0}(h^t) = 1 \).

Bond repayments are financed using bond issuances and lump-sum taxes. A government, which we describe later, collects taxes and issues riskless bonds.

2.2.2 Market Incompleteness

In our model, credit markets are incomplete along three dimensions, which correspond to the idiosyncratic risk, the aggregate one and credit constraints. Agents are assumed to trade only existing riskless zero-coupon, without being able to issue any financial security. This genders the three dimensional market incompleteness. First, no asset provides a payoff contingent to the idiosyncratic state of households. In consequence, the idiosyncratic risk uninsurable and markets are incomplete along this first dimension. Second, no security offers a payoff, which is contingent to the aggregate state of the economy. This assumption allows us to focus on the riskless yield curve, and builds up the third dimension of market incompleteness. Finally, agents are not allowed to issue any security and therefore face credit constraints. This property is the last incompleteness direction and is, with the idiosyncratic dimension, at the heart of a vast literature on liquidity constrained-economy. as named by Kehoe and Levine (2001) gave such a name to the literature launched by the seminal work of Bewley (1983) and studying economies with incomplete markets and credit constraints.

The various incompleteness dimensions does not matter identically for our purpose. It could be indeed easily possible to complete markets along idiosyncratic and aggregate risks. The parsimonious equilibrium, as well as our credit constraint and volume effects, would still remain. The difference is that it will not be riskless zero coupons, but assets contingent to idiosyncratic and aggregate states. What really matters for deriving our effects are credit constraints, which makes bond volumes have a wealth effect. As we will see later, when an agents falls into unemployment, he liquidates all his financial holdings (whatever their nature).
The asset volumes he liquidates thus influence the portfolio liquidation value and his wealth. At the difference of other incompleteness dimensions, credit constraints are crucial to derive our results. Both others help to concentrate on the riskless yield curve and not on Arrow Debreu assets.

2.3 Firms

The representative firm produces a single final good using a linear technology, with labor as single input. The labor productivity $z_t$ depends on the aggregate state of the economy. The productivity levels in state $h$ and $l$ are denoted respectively $z^h$ and $z^l$, with $z^h \geq 1 \geq z^l$. Firms’ profit maximization under perfect competition implies that the real wage $w_t$ is equal to the marginal productivity of labor, i.e. $w_t = z_t$.

2.4 Consumers

2.4.1 Preferences

Each agent has time-separable preferences defined over streams of consumption and labor. The single period utility is $u(c) - l$, where $c$ is the current consumption, $l$ the current labor supply, and $u$ is continuous, twice derivable with $u' > 0$, $u'' < 0$. The marginal disutility of labor is constant\(^4\) and normalized to 1. All agents discount their instantaneous utility with the same factor $\beta > 0$.

In period $t$, each agent $i$ consumes an amount $c^i_t$, supplies the labor quantity $l^i_t$, and demands the quantity $b^i_{t,k}$ of bonds of maturity $k$. The agent also pays a lump-sum tax $\tau_t(e^i_t)$, which is contingent to his employment status. Employed agents choose their labor supply and earn a hourly wage equal to $w_t$. Unemployed agents earn no labor income, but a constant amount of home production $\delta \geq 0$. Their labor supply is by definition equal to 0. We note $b^i_{t-1,k}$ the quantity of $k$–period bonds that the household $i$ holds at date 0.

\(^4\)Hansen (1985) justifies this assumption by the labor indivisibility. Scheinkman and Weiss (1986) first use it in incomplete market models.
2.4.2 Agent’s Program

The agent $i$’s problem consists in choosing the following streams of consumption, labor supply and bond holdings, respectively noted as:

\[
\begin{align*}
  c_i^t : & H^t \times E^t \to \mathbb{R}^+ \\
  l_i^t : & H^t \times E^t \to \mathbb{R}^+ \\
  b_{i,t,k}^k : & H^t \times E^t \to \mathbb{R}^+ & k = 1, \ldots, n
\end{align*}
\]

which maximize the household intertemporal expected utility:

\[
\sum_{t=0}^{\infty} \beta^t \sum_{h^t \in H^t} \nu_t(h^t) \sum_{e_{i,t} \in E^t} \mu_{i,t}(e_{i,t}) \left( u(c_i^t(h^t,e_{i,t})) - l_i^t(h^t,e_{i,t}) \right)
\]  

subject to the budget and non negativity constraints:

\[
\begin{align*}
  c_i^t(h^t,e_{i,t}) + \tau_t(e_{i,t}) + \sum_{k=1}^{n} p_{i,k}(h^t) b_{i,t,k}(h^t,e_{i,t}) &= \sum_{k=1}^{n} p_{i,k-1}(h^t) b_{i-1,k}(h^{t-1},e_{i,t-1}) \\
  &+ e_{i,t} z t l_i^t(h^t,e_{i,t}) + (1 - e_{i,t}) \delta \\
  c_i^t(h^t,e_{i,t}) &\geq 0 \\
  b_{i,t,k}(h^t,e_{i,t}) &\geq 0
\end{align*}
\]  

\[
\lim_{t \to \infty} \beta^t u'(c_t(h^t,e_{i,t})) b_{i,t,k}(h^t,e_{i,t}) = 0 \quad \text{for } k = 1, \ldots, n
\]  

The budget constraint (4) equalizes household’s spendings and revenues at date $t$. Incomes are the sum of the portfolio sale value and of labor earnings when $e_{i,t} = 1$, or of home production when $e_{i,t} = 0$. On the other side, the household chooses his current consumption and his saving in bonds of various maturities, and has to pay lump-sum taxes $\tau_t(e_{i,t})$. The constraint (6) stipulates that agents cannot issue debt securities and are thus borrowing constrained. The equation (7) is the set of transversality conditions, which always hold for the equilibrium we will consider.

Finally, we make the following assumption:

**Assumption B 1/z \leq u'(\delta)**

This assumption states that the marginal utility of the employed agent is larger than the unemployed one, meaning that the unemployed is worse-off than the employed one in all states.
of the world. The utility of one unit of labor paid at the lowest wage provides an upper bound for the jobless income $\delta$.

### 2.4.3 The Euler Equation

The Lagrangian function associated with the household $i$’s problem (3) expresses as follows:

$$L = \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in H^t} \nu_t \left( h^t \right) \sum_{e^{i,t} \in E^t} \mu_t \left( e^{i,t} \right) \times \left[ u' \left( c^i_t \left( h^t, e^{i,t} \right) \right) - l^i_t \left( h^t, e^{i,t} \right) + \sum_{k=1}^{n} \varphi_{t,k}^i \left( h^t, e^{i,t} \right) b^i_{t,k} \left( h^t, e^{i,t} \right) ight]$$

$$\delta \sum_{e^{i,t} \in E^t} \left[ e^{i}_t z^i_t l^i_t \left( h^t, e^{i,t} \right) + \left( 1 - e^{i}_t \right) \delta - \tau_i \left( e^{i}_t \right) + \sum_{k=1}^{n} p_{t,k-1} \left( h^t \right) b^i_{t-1,k} \left( h^{t-1}, e^{i,t-1} \right) \right]$$

$$- c^i_t \left( h^t, e^{i,t} \right) - \sum_{k=1}^{n} p_{t,k} \left( h^t \right) b^i_{t,k} \left( h^t, e^{i,t} \right) \right]$$

The Lagrange multipliers $\eta^i_t$ and $\varphi_{t,k}^i$ are positive functions defined over $H^t \times E^t$ and are respectively associated to the budget constraint (4) and to the borrowing constraints (6). The positivity of $c^i_t$ and $l^i_t$ in (5) are satisfied in the equilibrium we consider.

From the Kuhn and Tucker theorem, the optimality conditions are, for $t = 0, 1, \ldots$ and for all $(h^t, e^{i,t}) \in H^t \times E^t$:

$$u' \left( c^i_t \left( h^t, e^{i,t} \right) \right) = \eta^i_t \left( h^t, e^{i,t} \right)$$

$$\begin{cases} 
\eta^i_t \left( h^t, e^{i,t} \right) = 1/z^i_t & \text{if } e^{i}_t = 1 \\
l^i_t \left( h^t, e^{i,t} \right) = 0 & \text{if } e^{i}_t = 0 
\end{cases} \tag{9}$$

$$\eta^i_t \left( h^t, e^{i,t} \right) p_{t,k} \left( h^t \right) = \beta \sum_{h^{t+1} \in H^{t+1}} \nu_{t+1} \left( h^{t+1} \right) \sum_{e^{i',t+1} \in E^{i',t+1}} \mu_{t+1} \left( e^{i',t+1} \right) \eta^i_{t+1} \left( h^{t+1}, e^{i',t+1} \right) p_{t+1,k-1} \left( h^{t+1} \right)$$

$$+ \varphi_{t,k}^i \left( h^t, e^{i,t} \right) \quad \text{for } k = 1, \ldots, n \tag{10}$$

$$\begin{cases} 
\text{either } \varphi_{t,k}^i \left( h^t, e^{i,t} \right) > 0 & \text{and } b^i_{t,k} \left( h^t, e^{i,t} \right) = 0 \\
or \varphi_{t,k}^i \left( h^t, e^{i,t} \right) = 0 & \text{and } b^i_{t,k} \left( h^t, e^{i,t} \right) > 0 \quad \text{for } k = 1, \ldots, n 
\end{cases} \tag{11}$$

Eq. (8) defines household $i$’s consumption marginal utility, while Eq. (9) expresses the optimal labor supply. When he works ($e^{i}_t = 1$), the household equalizes the marginal gain in consumption of one unit of labor to its marginal disutility. Unemployed households do no supply any labor. Eq. (10) is the intertemporal optimality condition and it can be written more compactly as:

$$u' \left( c^i_t \left( h^t, e^{i,t} \right) \right) p_{t,k} \left( h^t \right) = \beta E_t \left[ u' \left( c^i_{t+1} \left( h^{t+1}, e^{i,t+1} \right) \right) p_{t+1,k-1} \left( h^{t+1} \right) \right] + \varphi_{t,k}^i \left( h^t, e^{i,t} \right). \tag{12}$$
\( E_t[\cdot] \) is the expectation over aggregate and idiosyncratic states, conditional on the information available at date \( t \) (here \( h^t \) and \( e^{i,t} \)). This standard Euler equation with credit constraints, equalizes the marginal cost of one unit of bonds of each maturity today to its marginal gain tomorrow. When the shadow cost of facing borrowing constraints is positive, meaning a binding credit constraint \( \varphi^i_{t,k}(h^t, e^{i,t}) > 0 \), the household \( i \) would increase his expected utility if he could issue \( k \)-period bonds. Eq. (11) explicits the relationship between the shadow-cost \( \varphi^i_{t,k} \) and the fact of facing borrowing constraints.

### 2.5 The government

There exists a government in the economy, whose unique role consists in balancing bond issuances and bond repayments. First, at each date \( t \), the government issues a given net quantity \( A_{t,k} \) of zero coupon bonds maturating in period \( t + k \). Bonds are sold at the market price \( p_{t,k}(h^t) \). Second, the government redeems bonds arriving at maturity at date \( t \), meaning all bonds of maturity \( k \) issued at date \( t - k \). We suppose that the government balances bond issuances and bond repayments using taxation, but the amount of debt issuances (i.e. \( A_{t,k} \)) is exogenous and arbitrarily.

In order to minimize tax distortions, the government is supposed to raise lump-sum transfers, contingent to the employment status. More precisely, we suppose that unemployed agents do not pay any tax\(^5\), whereas employed ones pay at date \( t \) the same amount \( \tau^e_t \). Lump-sum taxes, whose global amount reach \( \omega^e \tau^e_t \), adjust to balance the government budget constraint at each date \( t \).

\[
\sum_{k=1}^{\infty} p_{t,k} A_{t,k} + \omega^e \tau^e_t = \sum_{k=1}^{\infty} A_{t-k,k} \tag{13}
\]

The equation (13) is the government budget constraint of period \( t \), equalizing government resources and uses. The right hand side gathers resources, composed of taxes and debt issuances. These resources are used to redeem bonds arriving at maturity.

### 2.6 Bond Market Equilibrium

#### 2.6.1 Bond supply

The aggregate supply of securities for a given maturity is composed of newly issued bonds, but also of longer bonds issued earlier and becoming closer to maturity. At date \( t \), a quantity \( B_{t,k} \)

\(^5\)Introducing distorting taxes on labor or taxes on unemployed would imply redistributive and distorting effects, which are beyond the scope of this paper.
of bonds with maturity \( k \) is available on the market, with:

\[
B_{t,k} \equiv \sum_{j=0}^{n-k} A_{t-j,k+j}
\]

At date \( t \), the total supply of bonds with maturity \( k \) is composed of bonds issued in date \( t-j \) maturating \( j+k \) periods later. Their residual maturity is therefore equal at date \( t \) to \( k \).

As we study the equilibrium yield curve, we assume that the bond quantity for each maturity is constant \( B_{t,k} = B_k \), although the model can be easily extended to take into account stochastic changes in the bond supply. This assumption of constant stocks is equivalent to constant bond issuances: \( A_{t,k} = A_k \).

### 2.7 Market Clearing

The equilibrium condition on the bond market equalizes demand and supply for all dates \( t \)

\[
\int_0^1 b_{1,k}^1 (h^t, e_{i,t}) \, di = B_k \quad \text{for all } k = 1, \ldots, n
\]

In equation (14), the left hand side is the whole bond demand, equal to the sum of demands over all agents. The right hand side is the total exogenous supply.

### 3 A Limited Heterogeneity Equilibrium

In this section, we derive an equilibrium with limited heterogeneity, where we only have four agents. The derivation involves three steps. First, we conjecture the general shape of the solution; second, we identify the conditions, under which the hypothesized solution holds; and third, we show that these conditions are always fulfilled along the equilibrium we consider.

#### 3.1 Motivation

In general, heterogeneous agents’ models such as the one described above, generate an infinite-state distribution of agents’ types, as all individual characteristics (i.e. agents’ wealth and implied optimal choices) depend on the personal history of every single agent. In such cases, numerical methods are used to characterize the steady state distribution. In this paper, we derive a closed-form solution with a finite number of households types, thanks to two assumptions. The first one is the linearity of the utility function according to labor. This assumption
is used by Scheinkman and Weiss (1986) to simplify the behavior of employed households. The second one states that all unemployed households are borrowing constrained. In consequence, households sell their complete portfolio, as soon as they become unemployed. Such an assumption is used in a monetary setting by Kehoe, Levine and Woodford (1992).

3.2 Conjectured Equilibrium

We conjecture the existence of an equilibrium with only four agents’ types.

3.2.1 Technical assumptions

As we said before, we use two assumptions to reduce the heterogeneity: (i) the infinite elasticity in labor and (ii) the portfolio liquidation.

The infinite elasticity. This assumption states simply that agents have a linear disutility in labor. This assumption allows to reduce the heterogeneity amongst employed agents. From their optimality conditions (8) and (9), we deduce that their consumption $c^e_t$ is identical across employed households and only depends on preference parameters and the aggregate shock:

$$c^e = u'^{-1} (1/z_t) \ (> 0)$$

Employed agents equalize the marginal utility of consumption to the marginal pain of labor, which is supposed to be constant.

Even if employed agents all consume the same amount, they differ across their labor supply. Indeed agents, who were previously unemployed, do not hold any asset (cf. next assumption) and therefore have to work more to secure the same consumption level than agents, who were previously employed. Employed agents can therefore only be of two types: (i) the ee types, who are employed today and at the previous date, and (i) the ue types, who are employed today but jobless in the previous period.

One shot liquidation. We assume that there exists an equilibrium along which:

$$\begin{cases} \text{If } e^{i}_t = 1 & \text{then } \varphi^{i}_{t,k}(h^t, e^{i,t}) = 0 \\ \text{If } e^{i}_t = 0 & \text{then } \varphi^{i}_{t,k}(h^t, e^{i,t}) > 0 \end{cases} \text{ for all } k = 1, \ldots, n$$

(16)
This condition states that unemployed households \( (e^t_i = 0) \) does not hold any bond: \( b_{i,k}^t (h^t_i, e^t_i) = 0 \) for \( k = 1,\ldots,n \) if \( e^t_i = 0 \) (because of condition (11)). On the opposite, bond holders are always employed. As soon as an agent falls into unemployment, he liquidates all his financial positions; the condition insures that this is never sufficient to prevent him from being credit constrained\(^6\). We will further investigate the conditions under which such an equilibrium exists.

In consequence, as for employed agents, there are only two types of unemployed, depending on their previous employment status. If they were also previously unemployed, agents do not have any financial revenue and cannot invest in bonds. They consume their complete domestic endowment \( \delta \). On the opposite, if they were employed, they still cannot invest in bonds, but earn the liquidation value of their portfolio. In addition to their domestic endowment, they consume their financial earnings. In consequence, there only two types of unemployed agents: (i) \( uu \) type, who has been jobless since the previous period and (ii) \( eu \) type, who has just fallen into unemployment, but who was employed the date before.

**Reduced Heterogeneity.** Finally, due to both preceding assumptions, the economy is populated by only four agents’ types, that we note \( ee \), \( eu \), \( ue \), and \( uu \). They differ across consumption, labor supply and bond holdings. We are up to now only concerned with the consumption and the bond holdings. We postpone the labor supply investigation to the section, where we study the welfare properties of our equilibrium.

### 3.2.2 Agents’ consumption

We derive consumption level of our four agents in our reduced heterogeneity equilibrium.

**Employed agents.** The consumption level of employed agents (both \( ee \) and \( ue \)) has already been determined in equation (15):

\[
   c^e = u^{t-1} (1/z_t)
\]  

**Unemployed agents.** Unemployed agents do not consume the level, but it depends on their previous employment status. Agents \( uu \), who were previously unemployed only earn their domestic income \( \delta \), since they do not previously had access to the credit market. Moreover, they are currently borrowing constrained and consume their full endowment:

\[
   c^{uu}_t = \delta \ (> 0)
\]  

\(^6\) The period length can be defined such that the liquidation assumption is plausible.
An agent $i$ of $eu$ type, earns in addition to the domestic income $\delta$, the financial liquidation value of his bond portfolio. He is also credit constrained and consumes thus his complete endowment.

$$c_{i,eu}^t = \sum_{k=1}^{n} p_{t,k-1} b_{t-1,k}^i + \delta (> 0)$$ (18)

### 3.2.3 Agents’ bond holdings

We derive now the agents’ bond holdings. If this household is employed in the next period, which occurs with probability $\alpha$, then $\eta_{i+1}^t = 1$ (Cf. (9)). If the household falls into unemployment in the next period, then from (8) $\eta_{i+1}^t = u'(c_{t+1}^i)$. The agent is then an $eu$ type, whose consumption $c_{t+1}^{eu,i}$ is given by (18). From the first order conditions (8), (9), and (12) of the household program and the equilibrium condition (16), the Euler equation for employed households expresses as ( $k = 1, \ldots, n$):

$$p_t,k/z_t = \alpha \beta E_t \left[ p_{t+1,k-1}/z_{t+1} \right] + (1 - \alpha) \beta E_t \left[ u' \left( \sum_{j=1}^{n} p_{t+1,j-1} b_{t,j}^i + \delta \right) p_{t+1,k-1} \right]$$ (19)

We restrict our attention to the symmetric equilibrium\(^7\), where all employed agents hold the same quantities of bonds for maturities $k = 1, \ldots, n$. These quantities are only determined by preference parameters and aggregate variables. Hence, $b_{t,k}$ is the quantity of $k$–period bond held by employed agents at date $t$. All $eu$ agents now consume the same amount and the equation (18) simplifies into:

$$c_{t}^{eu} = \sum_{k=1}^{n} p_{t,k-1} b_{t-1,k}^{eu} + \delta$$ (20)

### 3.2.4 Market clearing

Because all employed households hold the same quantity of securities for each maturity, and unemployed households do not hold any asset, the financial market equilibrium (14) can be simplified. The global demand is thus equal to the constant demand $b_{t,k}^{e}$ of an employed agent weighted by the share $\omega^e$ (cf. (1)) of employed agents in the population.

$$\omega^e b_{t,k}^{e} = B_k$$ (21)

\(^7\)It is noteworthy that even in the general case, employed agents may hold different bond quantities, but the Euler equation remains, since the expected marginal liquidation value remains identical. What will differs is: (i) the labor quantities of employed agents and (ii) the consumption of $eu$ agents.
3.3 Pricing Equations

Using the market clearing condition (21), stating that all employed agents hold the same quantity of bonds, we derive the simple Euler pricing equations, which are the central equations of this paper. Using the Euler equation (19) and (21), one obtains:

\[ \forall t \geq 0 \quad \forall k \in \{1, \ldots, n\} \]
\[ \frac{p_{t,k}}{z_t} = \alpha \beta E_t \left[ \frac{p_{t+1,k-1}}{z_{t+1}} \right] + (1 - \alpha) \beta E_t \left[ \frac{1}{z_{t+1}} \sum_{j=1}^{n} p_{t+1,j-1} B_j / \omega \right] \quad (22) \]

The previous equations pin down the price of any bond as a function of the current and next aggregate states, of all future prices and of the bond supply. The price of a \( k \)-period bond expresses as the sum of two terms: (i) a smoothing one and (ii) a liquidation value, where a wealth effect intervenes. The first term reflects the standard consumption smoothing motive and values the bond through the marginal utility of an employed agent. This marginal utility depends on the wage rate, because the labor is endogenously supplied. The second term exists only if agents face credit constraints (\( \alpha < 1 \)). The bond is valued with the marginal utility of an unemployed agent, forced to liquidate his asset portfolio. Bond volumes, held by agents becoming unemployed, directly affect the portfolio liquidation value and thus equilibrium prices.

3.4 Analytical Bond Prices

3.4.1 Bond prices

We focus on the equilibrium, where bond prices depend only on the realization of aggregate shocks. Following the literature on asset pricing with finite state space (Mehra and Prescott, 1985 among others), we conjecture the following expression for bond prices:

\[ \forall t \geq 0, \forall k \in \{1, \ldots, n\}, \forall s \in \{h, l\}, \quad p_{t,k}^s = C_k^s z^s \quad (23) \]

This conjectured price structure exhibits a form of stationarity, as bond prices depend only on their maturity and on the current state of the world. In consequence, there are two yield curves, one for each value of the aggregate shock. The construction of the equilibrium below will prove the existence of this stationary equilibrium, when bond volumes are small.

The Euler pricing equations (22) expressed in both states \( h \) and \( l \) provide the expressions
for $C_k^h$ and $C_k^l$ for $k \leq 1$. We introduce the following notations:

$$\begin{bmatrix} C_h^0 \\ C_l^0 \end{bmatrix} \equiv \begin{bmatrix} 1/z^h \\ 1/z^l \end{bmatrix}$$

(24)

$$\begin{align*}
    u^h & \equiv u' \left( \delta + \sum_{i=0}^{n-1} C_i^h z^h B_{i+1}/\omega \right) \\
    u^l & \equiv u' \left( \delta + \sum_{i=0}^{n-1} C_i^l z^l B_{i+1}/\omega \right)
\end{align*}$$

(25)

The price structure (22) can be written compactly in a recursive form for $k = 1, \ldots, n$:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \beta T \cdot \begin{bmatrix} \alpha + (1 - \alpha) z^h u^h & 0 \\ 0 & \alpha + (1 - \alpha) z^l u^l \end{bmatrix} \cdot \begin{bmatrix} C_{k-1}^h \\ C_{k-1}^l \end{bmatrix}$$

(26)

This system provides $2 \times n$ equations that determine the $2 \times n$ coefficients $\{C_k^h, C_k^l\}_{k=1,\ldots,n}$. This system is not linear, because the whole price structure appears in coefficients $u^h$ and $u^l$.

### 3.4.2 Bond yields

The yield to maturity of a bond of maturity $k = 1, \ldots, n$ in state $s = h, l$ and in period $t \geq 0$ is defined by the standard expression $r_{t,k}^s = -\frac{1}{2} \ln p_{t,k}^s$. Interest rates are supposed to be continuously compounded. The average yield curve is the sum of yield curves in states $h$ and $l$ weighted by the average frequency of aggregate state $h$ and $l$ (given by the matrix $t$). The average yield $r_{t,k}$ of maturity $k$ at date $t$ expresses as:

$$r_{t,k} = \frac{1 - \pi^l}{2 - \pi^l - \pi^h} r_{t,k}^h + \frac{1 - \pi^h}{2 - \pi^l - \pi^h} r_{t,k}^l$$

(27)

This simple recursive structure will be extensively used below to derive properties of the yield curve. We first prove the existence of the equilibrium.

### 4 Existence of the equilibrium

The equilibrium existence is proved in four steps. We begin with deriving general conditions, under which our equilibrium with four agents' types exist. We then derive existence conditions with zero net supply and no aggregate shocks. We third prove continuity properties with respect to the variance of aggregate shocks and to the bond volumes. We finally derive the equilibrium existence conditions for a small variance of aggregate shocks and for small volumes.
4.1 General conditions for the equilibrium to exist

The stationary distribution with four agents’ types is constructed under the assumption that unemployed agents are borrowing-constrained. We now derive the conditions, under which this equilibrium exists.

4.1.1 Condition on the initial wealth

To avoid transitory dynamics, we assume that initial bond holdings of unemployed agents are 0, and that employed ones hold the same quantity \( b_{-1,k} \) of \( k \)-period bonds, where \( b_{-1,k} = B_k/\omega_k \).

4.1.2 Condition for the equilibrium existence

Our equilibrium exists if unemployed agents are credit constrained. It implies that the condition (16) is fulfilled for both \( eu \) and \( uu \) agents. This condition simplifies in two steps: (i) We use first order conditions (8), (9), (12), which express the conditions using consumption levels of various agents’ types, and (ii) we plug consumption level expressions (15), (17), and (20) into the preceding inequality.

For the agents \( uu \), we obtain the following equation, which states that the Euler equation for unemployed agents \( uu \) does not hold:

\[
\forall k = 1, \ldots, n, \quad p_{t,k} u' (\delta) > \beta (1 - \rho) E_t [p_{t+1,k-1}/z_{t+1}] + \beta \rho u' (\delta) E_t [p_{t+1,k-1}] \tag{28}
\]

The condition for unemployed agents \( eu \) is the same, except for the current consumption level, which is given by (20). We obtain then the following condition:

\[
\forall k \quad p_{t,k} u' \left( \delta + \sum_{j=1}^{n} p_{t,j-1} b_{t-1,j}^e \right) > \beta (1 - \rho) E_t [p_{t+1,k-1}/z_{t+1}] + \beta \rho u' (\delta) E_t [p_{t+1,k-1}] \tag{29}
\]

Because agents hold a positive quantity of assets, unemployed agents \( uu \) without any financial holding are borrowing constrained as soon as newly unemployed agents \( eu \), who benefit from their past bond holdings, are. The condition (28) is therefore fulfilled as soon as the condition (29) is. In consequence, we only have to check that agents, who fall into unemployment and decide to liquidate their portfolio are always credit-constrained.
4.2 Equilibrium existence with zero net supply and no aggregate shock

Using the zero net supply assumption and prices given by (26) in the condition (29), credit constraints bind for the unemployed agent if:

\[(\alpha + (1 - \alpha)u'(\delta))u'(\delta) > (1 - \rho) + \rho u'(\delta)\]

Because \(u'(\delta) > 1\) (assumption B), this condition is fulfilled\(^8\) as soon as \(\alpha < 1\). In zero net supply, unemployed agents are always credit constrained, when there are idiosyncratic shocks.

4.3 Continuity of the yield curve as a function of bond supply and shocks

The following proposition summarizes the regularity property of the yield curve. We will use it extensively below to generalize results from the zero net supply to the (small) positive supply. We introduce the following notations: \(B\) is the (column) vector of bond quantities for the \(n\) maturities: \(B = [B_1 \ldots B_n]^\top\), \(Z\) the vector of wages (or equivalently productivities) \(Z = [z^l \ z^h]^\top\) and \(C\) is the vector of coefficients for both states \(h\) and \(l\) and the \(n\) maturities:

\[C = [C^h_1 \ C^h_2 \ldots C^h_n \ C^l_1 \ C^l_2 \ldots C^l_n]^\top\]. \(1_n\) and \(0_n\) are vectors of length \(n\) containing only resp. 1 and 0.

**Proposition 1** If \(B\) is in the neighborhood of \(0_n\) and \(Z\) in the neighborhood of \(1_2\), then \(C\) is a \(C^1\) function of \(B\) and of \(Z\).

The proof of the proposition is left in appendix. This proposition states simply that the introduction of positive bond quantities as the aggregate risk do not make the yield curve jump. Bond prices are a continuously derivable with respect to the bond volumes, as well as aggregate productivity.

4.4 Equilibrium existence in the general case

We use the proposition (1) on the regularity of \(C\) to generalize the equilibrium existence to the positive supply case. The system (26) with initial conditions (24) defines the vector \(C\) as a continuous function of \(B\) and \(Z\), when \([B^\top \ Z^\top]\) is in a neighborhood \(V_1\) of \([0_n^\top \ 1_2^\top]\). Moreover,

\(^{8}\)The RHS reaches its maximum \(u'(\delta)\) when \(\rho = 1\) and when \(\alpha < 1\), we have \((\alpha + (1 - \alpha)u'(\delta))u'(\delta) > u'(\delta)\).
the equilibrium exists in the zero net supply without aggregate shocks: If $[B^\top Z^\top] = \begin{bmatrix} 0_n^\top & 1_2^\top \end{bmatrix}$, the equilibrium vector $C$ satisfies conditions (29). By continuity, there exists a neighborhood $V_2 \subset V_1$ of $\begin{bmatrix} 0_n^\top & 1_2^\top \end{bmatrix}$, such that conditions (29) is fulfilled if $[B^\top Z^\top] \in V_2$.

If the supply of bonds for any maturities remains small, and if the variance of the aggregate shock is low enough (the process $z$ remains around the mean 1), then the equilibrium with four agents’ types exists.

5 A Simple Model

We simplify the preceding model to shed light on yield curve implications in a straightforward framework. We make the following assumptions:

(i) Agents can only invest in two securities, whose mature respectively in one and two periods.

(ii) The aggregate risk $z$ is an i.i.d. process equal to $z^h = 1 + \varepsilon$ with probability $1/2$ and $z^l = 1 - \varepsilon$ with probability $1/2$. The variation $\varepsilon$ is small, so that we focus only only on second order approximations in $\varepsilon$.

(iii) The utility is quadratic and the marginal utility is linear in consumption: $u'(c) = u_1 - u_2 c$ with $u_1, u_2 > 0$. Constants $u_1$ and $u_2$ are such that $u$ fulfills the standard assumptions regarding growth and concavity of the utility function. The explicit conditions on $u_1$ and $u_2$ are given in appendix.

Preceding equations as well as the Euler equation (22) provides the following expressions for bond prices $p_{1,t}$ and $p_{2,t}$, which satisfy the following equations:

\[
p_{1,t}/z_t = \alpha \beta E_t \left[ z_{t+1}^{-1} \right] + (1 - \alpha) \beta E_t \left[ u' \left( \delta + B_1 + B_2 p_{1,t+1} \right) / \phi \right] \quad (30)
\]

\[
p_{2,t}/z_t = \alpha \beta E_t \left[ p_{1,t+1}/z_{t+1} \right] + (1 - \alpha) \beta E_t \left[ p_{1,t+1} u' \left( \delta + B_1 + B_2 p_{1,t+1} \right) / \phi \right] \quad (31)
\]

As in the general model, both prices are proportional to the level of aggregate shock: $p_{t,t} = C_t z_t$. As $Ez_t = 1$, the average prices of both bonds are: $p_1 = E p_{1,t} = C_1$ and $p_2 = E p_{2,t} = C_2$. After a seocd order approximamtion in $\varepsilon$, price equations (30) and (31)
simplify into:

\[
\begin{align*}
    p_1 &= \frac{\alpha \beta \left(1 + \varepsilon^2\right) + (1 - \alpha) \beta (u_1 - u_2(\delta + B_1))}{1 + (1 - \alpha) \beta u_2 B_2} \\
    \frac{p_2}{p_1} &= \alpha + (1 - \alpha) \beta \left(u_1 - u_2 \left(\delta + B_1 + p_1 B_2 \left(1 + \varepsilon^2\right)\right)\right) / \phi
\end{align*}
\]

Let us comment the preceding price expressions.

First, without credit constraints (\(\alpha = 1\)), bond prices remain unchanged, whatever their volumes\(^9\). However, as soon as credit constraints can really bind (\(\alpha < 1\)), bond volumes affect both prices through two channels. The first one is a wealth effect, which diminishes both prices. For the agent who falls into unemployment, a larger bond volume increases his portfolio liquidation value, simply because he sells a larger quantity of securities. Since they will be wealthier when falling into unemployment, the self-insurance motive softens. The bond demand decreases and so do prices. The second channel affects only the long term bond, through the price of the short bond \(p_1\). Whereas the short bond yields 1 unit of goods in the next period, the long term bond is sold at the uncertain price \(p_1\), which diminishes with larger bond volumes. This price effect diminishes the demand for the long term bond, which will be sold tomorrow at a lower price. This mechanism also contributes to decreases the long term bond price.

Because of both preceding channels, bond volumes do not generally affect identically both prices. In order to investigate the relative impact of bond volumes on prices, we focus now on the slope of the curve, when the probability of becoming credit constrained is low (\(\alpha\) close to 1). It appears to be the relevant case for all developed countries\(^10\). The average slope \(S\) of the yield curve expresses as \(S = -\frac{1}{2} \log \frac{p_2}{p_1}\). After some calculations, the slope expression simplifies to:

\[
2 S/\varepsilon^2 = 1 - (u_1 - u_2 (B_1 + \delta) - 2 u_2 B_2 \beta) (1 - \alpha) + O (1 - \alpha)^2 + O(\varepsilon^2) \quad (32)
\]

From the expression (32), we deduce unambiguously that a larger bond volumes, whatever their maturity, steepens the yield curve. Two different mechanisms are at stake to explain the respective roles of short volumes and long ones. The intuition for long bonds is similar to the

\(^9\)This result stems from our general equilibrium model. In an endowment economy as in Lucas citeyearLu:78, volumes would affect prices because the whole economy becomes wealthier when bond volumes go up (as in Piazzesi and Schneider, 2007).

\(^10\)Several authors have advocated the use of job-loss probabilities, rather than income fluctuations per se, to proxy for the idiosyncratic risk faced by individuals (e.g., Carroll et al. (2003)). Using yearly job-loss probabilities from the data in Nickell et al. (2001), we find that the probability of remaining employed is always higher than 0.96 % annually. The probability to stay employed is even higher for a higher frequency.
preceding explanation: The long run securities bought in a given period are sold in the next period at the uncertain price \( p_1 \). Long term bonds thus carry a specific risk of liquidation, which means a risk premium. If the volume \( B_2 \) of long term bonds is large, this risk concerns a large number of securities and thus commands a high premium of long run titles over short run ones. When volumes enlarge, long assets become relatively cheaper than short ones, which makes the yield curve steepen.

The mechanism for \( B_1 \) (or equivalently \( \delta \)) is slightly different. An increase in \( B_1 \) makes the liquidation value of the portfolio larger in all states of the world and therefore diminishes the demand for self-insurance and thus make all assets cheaper. However, this fall in prices affects more long assets, because they are riskier than short ones. They will be sold next period at a smaller price \( p_1 \). It becomes less favorable to sell this asset in a bad state of the world (in case of portfolio liquidation or not). Long assets command then an increased risk premium and become cheaper than short ones.

We have shown in this simple model how bond volumes are likely to affect the level and the slope of the yield curve. In the following, we generalize these results to an undefined number of maturities and discuss their relation to the literature.

6 Yield Curve Implications

In this section, we investigate yield curve implications of our model. First, we show how credit constraints affect the shape of the yield curve. Second, we perform a similar exercise for the impact of bond volumes. Third, we prove to what extent our model contributes to reject the Expectation Hypothesis. Finally, we use a Pareto criterion to rank our equilibria according to bond volumes.

Before that, we show that the yield curve in good state of the world lies uniformly below the one in bad state.

Lemma 1 (Yield curve ranking) If \( B \) is small enough, then the yield curve in good state of the world lies strictly below the one in bad state. \( r^h_k < r^l_k \) \( k = 1, \ldots, n \). Yields in both states of the world converge to a common limit: \( \lim_{k \to \infty} r^l_k = \lim_{k \to \infty} r^h_k = r^{\lim} \).

This result does not specifically rely on borrowing constraints (See Davidson et al. 1990 and Seppälä (2004) for some evidence on the real yield curve in UK). If the productivity is high, agents have indeed more wealth to save and desire to transfer resources in less favorable
states of the world, when the wage is lower. This larger demand for bonds raises prices and thus shifts globally the yield curve downwards.

The long term yield in both states of the world in common, since the corresponding bond payoffs in an average state of the world.

Using this common long term yield together with the short one \( r_1 \) given by (27), we can define the average slope \( \Delta \) of the curve, that we will investigate in the next paragraphs: \( \Delta = \lim r_{\text{lim}} - r_1 \).

We now turn to the yield curve implications of our model.

### 6.1 Credit Constraint Effects

We first derive the effect of facing credit constraints, when bonds are in a given supply. The focus on volumes is postponed to the next section. The following proposition summarizes the credit constraint effects on the shape of the curve.

**Proposition 2 (Credit contraint effects)** If bond volumes \( B \) are small enough and credit constraints are binding \( (\alpha < 1) \):

1) The level of the curve decreases with credit constraints.

Moreover, if the probability of facing credit constraint is not too high \( (\alpha \) close to 1):

2) The slope of the curve decreases with credit constraints.

3) The variance of all yields \( r_k \) increases with credit constraints, \( k = 1,\ldots,n \).

The first result proves that the average yield curve shifts downwards, when agents are more likely to be credit constrained. Tighter credit constraints raises forces agents to self-insure themselves more, which puts the bond demand up. Prices increase and yield diminish. The result 1) extends to the whole yield curve a result known since Huggett (1993). Borrowing constraints may help to explain the low level of interest rates, relative to the predictions of complete market models. Incomplete markets are therefore likely to solve the so-called risk free rate puzzle (Weil (1989)).

According to the third result, the variance yields increases with credit constraints. If the demand for self-insurance widens for more probable credit constraints, the impact is not identical is all states of the world. In a good state, agents are richer than a bad one, which puts
the asset demand up: The higher the productivity, the larger the demand for bonds. Prices increase thus more in good states that in bad ones, which enlarges the gap between yield curves in both states of the world. In consequence, borrowing constraints contribute to increase the interest rates volatility. Hornstein and Uhlig (2000) points out that this volatility is too low in standard complete market RBC models. Credit constraints are a possible channel to overcome this issue.

We have presented presented results concerning the effect of credit constraints on the yield curve. We now turn to the impact of bond volumes on the shape of the yield curve.

### 6.2 Volume effects on the yield curve

The following proposition summarizes the bond volume effects on the yield curve.

**Proposition 3 (Bond volume effects)** If $B$ is small enough and and credit constraints are binding ($\alpha < 1$):

1) A larger bond volume, whatever the maturity, increases the level of the curve: $\frac{\partial p_i^k}{\partial B_i} < 0$ for $i, k = 1, \ldots, n$ et $s = h, l$.

   Moreover, if the probability of facing credit constraint is not too high ($\alpha$ close to 1):

2) A larger bond volume, whatever the maturity, increases the slope of the curve. $\frac{\partial \Delta}{\partial B_i} > 0$ for $i = 1, \ldots, n$.

3) The variance of all yields $r_k$ decreases with the bond volumes $B_i$, whatever the maturity.

This proposition explains how the change in bond supply affects the shape of the yield curve. Effects only happen, when credit constraints can really bind ($\alpha < 1$). Without credit constraints ($\alpha = 1$), bond prices remain unchanged whatever their volumes, which is the standard Ricardian result. Other papers study bond volumes in partial equilibrium models à la Lucas (1978). In this case, a larger bond volume increases the resources available in the economy, which affects the aggregate wealth and prices. (as in Piazzesi and Schneider (2007)).

We propose a general equilibrium model, where the bond supply is financed by taxes and does not affect the aggregate wealth.

According to the point 1), a larger bond supply (whatever its maturity) decreases bond prices for all maturities in all states of the world, even for long run prices. Agents $eu$, who liquidate their portfolio, are richer when bond volumes are higher. In consequence, they demand
less bonds of any maturity to self-insure themselves against the unemployment risk. This smaller demand lowers prices and the yield curve shifts then upwards. The positive effect of bond volumes on interest rates levels has been underlined in various empirical works (Duffee, (1996); Fleming, (2002) among others). It is noteworthy that our model predicts that a larger volume of a single bond (whatever its maturity) affects the whole yield curve in each aggregate state of the world.

In order to investigate the relative impact of bond volumes on prices, the point 2) focuses on the slope of the curve, when the probability of becoming credit constrained is low (\( \alpha \) close to 1), what seems, as we said before, the most relevant case for developed countries. A larger bond volume, whatever its maturity, steepens the yield curve. This larger volume makes indeed the portfolio liquidation value more volatile: The agent has a larger quantity of assets to sell at an uncertain price. Long run assets become then relatively riskier than short ones, because they are more likely to be liquidated: The longer the maturity, the more probable the liquidation at an uncertain price. The agent demands thus a larger risk premium for long assets, relative to short ones, which steepens the curve.

The third and last point proves that larger bond volumes reduce the volatility. It indeed decreases prices in both states of the world, but more in good states of the world, when agents are richer and have a smaller marginal utility. A larger bond volume diminishes the difference between bond prices in both states of the world, and so does the variance.

### 6.3 Expectation Hypothesis

#### 6.3.1 Definition

In this section, we investigate to what extent our model is able to reject the expectation hypothesis (EH). The expectation hypothesis can be expressed in various equivalent ways. The most standard definition is to suppose that the \( n \) period yield is the sum of expected future short yields, up to a constant, which may depend on the yield maturity but not on time. We note \( \theta \) the risk premium associated to the yield \( r_{t,n} \) at date \( t \) of a zero-coupon of maturity \( n \).

---

Other empirical studies on the effect of volumes on prices have found the opposite effect (Amihud and Mendelson, (1991)). This paper focuses on markets, which are not very deep. According to Jegadeesh (2002), this contradictory result stems from a focus on markets, which lack depth. In this case, larger volumes raise trading volumes, which diminishes trading frictions and raises the price. We do not model here such a microstructure argument.
The expectation hypothesis (EH) has the following signification:

$$r_{t,n} = \frac{1}{n} \sum_{k=0}^{n-1} E_t r_{t+k,1} + \theta^n$$

(33)

This definition is equivalent to:

$$(n - 1)E_t(r_{t+1,n-1} - r_{t,n}) = \delta_n + (r_{t,n} - r_{t,1})$$

$$\delta_n = \theta^n + (n - 1)(\theta^n - \theta^{n-1})$$

Testing the EH consists simply in running the following regression, for all maturities $n \geq 2$:

$$r_{t+1,n-1} - r_{t,n} = \alpha_n + \beta_n \frac{r_{t,n} - r_{t,1}}{n-1} + \varepsilon_{t,n}$$

(EH)

The EH leads to $\alpha = \delta_n$ (and equal to 0 without risk-premium) and $\beta_n = 1$ for all maturities. This last equality stipulates that the current slope of the yield curve is a perfect unbiased predictor of the future rates, which is equivalent to constant term risk premia. Campbell and Shiller (1991), followed by many others, run this regression on US nominal data for several maturities and report that coefficients were significantly smaller than 1 and even negative. Moreover, they display a downward sloping pattern. Seppälä (2004) performed a similar regression with UK real data and found that the coefficient $\beta_n$ was below 1 and decreasing, at least for the first maturities.

### 6.3.2 The expectation hypothesis with incomplete markets

To prove that incomplete markets are helpful in rejecting the EH, we focus on the coefficients $\beta_n$ of the expectation hypothesis regression (EH). We notably show that it is below 1 and that it decreases with credit constraints. The coefficient $\beta_n$ expresses as:

$$\beta_n = (n - 1) \frac{\text{Cov}(r_{t+1,n-1} - r_{t,n}, r_{t,n} - r_{t,1})}{\text{Var}(r_{t,n} - r_{t,1})}$$

The following proposition summarizes our results regarding the expectation hypothesis.

**Proposition 4 (Rejection of the Expectation Hypothesis)** If the good state of the world is more likely to occur than the bad one ($\pi^h > \pi^l$):

1. For every maturity $n$, the coefficient $\beta_n$ decreases with credit constraints ($\frac{\partial \beta_n}{\partial \alpha} > 0$).
2. In presence of credit constraints \((\alpha < 1)\), and under two technical conditions, the coefficient \(\beta_n\) is strictly smaller than 1 and decreases with \(n\): \(\beta_{n+1} \leq \beta_n \leq 1\)

The proposition states that facing credit constraints is able to gender the departure from the EH. In the appendix, we prove that the coefficient \(\beta_n\) can be strictly smaller than 1 and decreasing with \(n\), depending on the value of the parameters.

The condition \(\pi^h > \pi^l\) requires that the good state is more persistent than the bad one. Hamilton ((1994), chap.22) shows that it is the case for the US. Contractions typically persist for 4 quarters, whereas expansions last 10 quarters on average. In our economy, it implies that the yield curve in the good aggregate state occurs more often than the one in the bad state. The properties of the average yield curve are thus close the ones of yield curve in the good state of the world.

The departure from the EH is the result of the wealth effect in the Euler equation, which stems from the uninsurable liquidation risk.

### 6.4 Welfare implications

We characterize using a Pareto welfare criterion, how bond volumes impact our equilibria. In our model, the size of bond issuances is arbitrarily and taxes adjust, so as to balance bond issuances and bond repayments. Our taxation scheme (lump-sum taxes concerning only employed agents) minimizes distortions and only impacts the labor quantity of employed agents.

For any size issuances, our equilibrium indeed exists, as long as bond volumes are small enough to insure that unemployed agents are credit constrained. The following proposition ranks equilibria according to a Pareto criterion, with respect to the bond supply.

**Proposition 5 (Pareto ranking of equilibria)** Without aggregate shock, a larger bond supply (but small enough to have the equilibrium existence):

1. always increases, in a Pareto sense, the ex-ante welfare,

2. always increases (Pareto) the welfare of ee and eu agents, but increases the welfare of ue and uu agents if and only if \(\beta > [\alpha + (1 - \alpha)u]^{-1}\).

This proposition sums up two results. First, from an ex ante point of view, when agents ignore their types, a larger bond supply is Pareto improving. Second, ex-post, this improvement is not homogeneous across agents' types. Agents, who were previously employed and thus, who can sell bonds at the beginning of the period, benefit from the larger bond volumes. The
financial wealth increase more than offsets the higher tax. On the opposite, agents, who do not have any financial holding at the beginning of the period suffers from the larger bond volumes, because of a tax raise.

More precisely, the instantaneous utilities of agents ee and uu are not affected. It is neutral for uu since they only earn a constant income $\delta$. For ee agents, the increase in taxation (which finances a larger bond supply) is strictly\textsuperscript{12} compensated by the higher sale value of their portfolio. On the opposite, types ue do not have any asset to sell (they were unemployed and thus credit constrained in the preceding period) and only sustain the tax increase, which makes them work more. Finally, agents eu are unemployed, but do not pay any tax: They therefore benefit from a higher value of their bond portfolio. Instantaneously, a larger bond supply is a redistribution from ue (who work but do not have any security for sale) to eu (who are unemployed, but liquidate a bond portfolio). Since the marginal gain of eu is larger than the marginal loss of ue, and since both types are equally probable, the first part of the proposition is straightforward: ex ante, a larger bond supply is always Pareto improving.

The ex post welfare comparison from date 0 point of view, when agents know their type, is less direct. The expected utility for each type balances today’s impact and tomorrow’s one. It is noteworthy that the situation of the agent ee (resp. uu) is analogous to the one of eu (resp. ue), since his current welfare is not impacted (resp. impacted) by bond supply. On the one hand, the ue agent’s utility is positively affected by a larger bond supply, only if his current loss equalizes the gain of becoming possibly tomorrow eu: If the agent is patient enough, the increase in bond supply will be welfare improving (if not, he will suffer from a decrease in welfare). On the other hand, the today’s gain of the agent eu is mitigated by the probability of becoming tomorrow ue, and therefore by the fact of possibly suffering from a larger bond supply. However, instantaneous utilities imply that this possible loss cannot be large enough to offset the today’s certain gain. Finally, eu agents, as ee ones, always benefit from an increase in bond supply, whereas ue and thus uu may suffer from it, if they are not patient enough.

7 Quantitative exploration of the model

We investigate the empirical yield curve movements of our model. Before empirical results, we present our calibration strategy.

\textsuperscript{12}Complete calculation are available in appendix.
7.1 Model calibration

In order to assess the empirical behavior for our model, we need to calibrate four types of parameters: (i) composition of the population, which characterizes the model heterogeneity, (ii) the preference parameters of all individuals, (iii) the wage process and (iv) the bond supply.

For all data, our time step is the quarter.

**Heterogeneity.** We use Imrohoroglu (1992) values to calibrate the model heterogeneity and in particular the matrix $t$ driving the idiosyncratic shock of unemployment. However, since her time baseline is 6 weeks, we need to convert her data into quarterly ones. We obtain then:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\alpha$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>0.9366</td>
<td>0.2717</td>
</tr>
</tbody>
</table>

**Preference parameters.** We suppose that the consumption utility is log, so that the period utility of a given agent consuming $c$ and working $l$ is:

$$u(c, l) = \ln(c) - l$$

**Wage process.** We calibrate directly the wage process $z$ using the NIPA (National Income and Product Accounts) tables from the Bureau of Economic Analysis (BEA). In order to focus explicitly on the wage, we use the Wage and salary disbursements in the private industry. We deflate the data using the consumer price index and compute the real wage in the private industry by capita. The population figures come from the US Census Bureau. Data cover the period from Q1 1947 to Q2 2007.

We assume that at each date, the real wage per capita expresses the product of a structural term and a cyclical one. We compute the structural term using the Hodrick and Prescott filter (1981) with a penalty of 1600, which is standard for quarterly data. We finally focus on the cyclical term $z_t$. It is supposed to follow a two state Markov process, equal to either $z^h$ or $z^l$, with the transition matrix $T$.

$$z_t = z^{s_t} + \varepsilon_t$$

$$s_t = \begin{cases} h, \\ l \end{cases}$$

with the transition matrix $T$ defined in (2)

$$\varepsilon_t \sim N(0, \sigma^2)$$
We use Hamilton (1989) procedure to compute the likelihood of this simple Markov switching process. Its maximization leads to the following results:

<table>
<thead>
<tr>
<th></th>
<th>$z^h$</th>
<th>$z^l$</th>
<th>$\pi^h$</th>
<th>$\pi^l$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1.015</td>
<td>0.982</td>
<td>0.925</td>
<td>0.907</td>
<td>0.0142</td>
</tr>
<tr>
<td>Std Err.</td>
<td>0.001</td>
<td>0.001</td>
<td>0.037</td>
<td>0.042</td>
<td>0.001</td>
</tr>
</tbody>
</table>

In good times, the cyclical component is 1.5% higher than the trend. The probability that a good quarter follows another good quarter is 0.98. A wealthy period persists on average for $1/(1 - 0.98) \approx 55.5$ quarters, whereas a periods below trend persists for 13.3 quarters. Hard episodes are typically 1.8% below the wage trend.

Both types of episodes are highly persistent and our assumption A regarding probabilities $\pi^h$ and $\pi^l$ is largely fulfilled: $\pi^h + \pi^l = 1.91 >> 1$. The assumption in the proposition 4 is also valid since $\pi^h > \pi^l$.

**Bond supply.** Our model is essentially a real one and does not account for nominal assets. However, we need a proxy for (i) the amount of bond holdings by households and (ii) their maturity structure. For the amount, we use the Flow of Funds Accounts provided by the Federal Reserve. The balance sheet of households in Q2 2007 states that households hold 3244.6 billions of dollars in credit market instruments, which include various types of bonds, like Treasury securities, municipal securities, corporate and foreign bonds. At the same date, the personal income of households reach $11,619 bn. It means that bond holdings correspond roughly to 28% of the quarterly disposable income.

However, this figure is silent about the maturity structure of household holdings. As a proxy, we use marketable public debt maturities, because we do not have any data concerning the maturity structure of corporate bonds. It is nonetheless noteworthy that the maturity structure has little quantitative impact on our results. More precisely, the US treasury debt service publishes each month a *Monthly Statement of the Public Debt*\(^{13}\) and we focused on the one of September 2006. This file provides a picture at a given date of the situation of the public debt. In particular, for the public debt, one can find for T–Bills, T–notes and T–bonds, the outstanding amount, the coupon, the issue date and the maturity. We have transformed these raw data into zero coupon bonds using times-to-maturity, outstanding amounts and coupons. We obtained finally a profile of bond supply per maturities. Since we only focus on the 10 first years of the yield curve, we gathered all amounts after 10 years in the last one.

\(^{13}\)http://www.treasurydirect.gov/govt/reports/pd/mspd/mspd.htm
Finally, we have been silent about a last parameter $\beta$, which is the level of the yield curve. As is standard in this literature which departs from perfect financial markets, we calibrate it to get a reasonable value for the level of the yield curve with bond supply, which leads to $\beta = 0.985$. This is very close to the standard quarterly $\beta$ value of 0.999.

### 7.2 Yield curve simulation

The following graphs represent the yield curve as well as the variance as function of the maturity. Every one of them plots three lines, which describe exactly the same model with the same parameters except for bond volumes and credit constraints. One of the curve reflects the zero net supply case, another one the zero credit constraint case\(^{14}\) and finally the positive supply case, with credit constraints.

The figure (Fig. 1) plots the average yields $r_k$ as a function of the maturity $k$, for maturities varying between 1 and 40 quarters. The three lines correspond respectively to the zero net supply case, to the zero credit constraint case, and to the volume effects one. This graph is a relevant illustration of our propositions 2 and 3.

![Impact of bond volumes on average yield curves](image)

**Figure 1.** Average Yield Curves

The graph (Fig. 1) illustrates, to what extend the shape of the yield curve modifies with bond volumes, when they vary from zero (zero bond supply) to a ‘large’ amount (zero credit constraints).\(^ {14}\) In this case, bond volumes do not play any role.
It is noteworthy that our empirical exercise is mainly illustrative. We have not tried to estimate the yield curve and to reproduce the ‘true’ shape for the real curve. There are at least three reasons for that. First, as our goal is to exhibit new channels, we kept away from extensions in several dimensions, which could be useful to confront the model to the data. Second, data available on inflation-linked bonds do not allow to build up a consensus regarding the true shape of the curve, at least in the UK market. We do not want to take part in it. Finally, the macro process (Wage and salary disbursements in the private industry) used for the calibration does not exhibit a sufficient variation to allow us to replicate properly market data, and especially the slope. Our figures illustrate however the variety of shapes it is possible to obtain with this model. It is notably able to replicate a downward sloping curve.

The graph (Fig. 2) plots the curves in the zero and positive bond supply cases, with the true scales. It appears, that the slope effect is in terms of magnitude a secondary effect relative to the level effect. With positive bond volumes, the long term yield jumps from $-10\%$ (bonds are so scarce that agents are likely to pay a price greater than 1) to $1.2\%$.

![Average yield curves (rescaled)](image)

**Figure 2.** Yield Curves with true scales

The last figure (Fig. 3) draws the variances of yields $\text{Var}(r_k)$ as a function of the maturity,
in the three cases: no bond, no credit constraint, and calibrated model. In all cases, the variance is a decreasing function of the maturity.

\[ x \times 10^{-4} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{yield_variances.png}
\caption{Yield Variances}
\end{figure}

8 Empirical Investigation

In this section, we measure to what extent our model predictions are reflected in the data. More precisely, we check out that interest rates exhibit properties that we have proved before. First, larger bond volumes raise yields and steepen the yield curve. Second, tighter credit constraints diminish yields and the slope of the curve. Using a panel data of OECD countries, we show that rates actually verify these properties. Before presenting our estimation strategy and empirical results, we describe our dataset.

8.1 The Dataset

We use a panel dataset of 17 OECD countries, with both macro and financial variables. We consider only annual data, from 1983 to 2005.
8.1.1 The Variables

We consider both macro variables and financial variables, in a yearly basis. The macro variables include the real GDP growth, the inflation (yearly change in consumer/retail price index), the standard unemployment rate, and the gross public debt as a fraction of GDP. The financial variables include the nominal short- and long-term interest rates. All these data come from the OECD or the AMECO databases. Both are available online\(^\dagger\). As stated in the website preamble, AMECO is the annual macro-economic database of the European Commission’s Directorate General for Economic and Financial Affairs (DG ECFIN). It gathers data for EU-27 countries as well as other OECD countries: United States, Japan, Canada, Switzerland, Norway, Iceland, Mexico, Korea, Australia and New Zealand. The OECD database covers a large range of statistics for all OECD countries.

More precisely, we derive the real GDP growth from the gross domestic product at constant market prices. We use the general government consolidated gross debt to compute the debt as a share of GDP. The unemployment rate relies on the Eurostat definition. The nominal short-term interest rate is roughly the 3-month market interbank rates and is very close to the government 3-month rate. We derive the short real rate by subtracting the inflation at the next period, which approximates the expected inflation. The long-term rate is extracted from prices of long-term government bonds. The precise maturity of these bonds may vary across countries, because of different domestic bond markets and also across time, due to the development of domestic bond markets. However, for most countries, this long-term rate is the 10 year yield of the corresponding government bond. The real long term rate is also computed by subtracting the expected inflation. All these data come from the AMECO database, except the inflation. The inflation is the percentage change in one year of consumer price index and comes from the OECD Main Economic Indicators (MEI) database. It notably presents the characteristics to be comparable on an international basis.

In addition to these macroeconomic variables, we need to obtain the probability \(1 - \alpha\) of being credit constrained. Unfortunately, such data are not available and we need to find a proxy for it. Following our model, as well as Carroll et al. (2003), we consider job-loss probabilities to measure the probability of facing credit constraints. The annual job-loss probability is the average of monthly probabilities. Each monthly probability is equal to the number of unemployed for less than one month divided by the civilian employment population. Those

\(^\dagger\)The AMECO website is http://ec.europa.eu/economy_finance/indicators/annual_macro_economic_database/ameco_en.htm and the OECD one is http://stats.oecd.org/ubos/default.aspx.
data are notably available through the OECD and we have partially extended them backwards using Nickell et al. (2001) data. We use the probability of falling into unemployment during the month as a proxy for the probability of being credit constrained. As we will see later, this makes us face a couple of issues.

Finally, we obtain a panel dataset with 17 countries and 23 years, from 1983 to 2005. However, due to missing data, we do not have a balanced panel dataset and have 370 observations on the whole. It is possible to restrict further the number of countries and the horizon to get a balanced dataset, but it does not change qualitatively the results.

8.1.2 Empirical Strategy

We want to measure to what extent both the level and the slope of the curve react to tighter credit constraints and larger bond volumes. We proceed in two steps: (i) a naive estimation, with and without fixed effects for the country and the year, (ii) the instrumentation of the job-loss probability. This last regressions validate our theoretical predictions: Tighter credit constraints decrease both the level and the slope of the curve, whereas larger bond volumes increase them.

We approximate the level of the curve with both the real short rate and the real long rate, that we note resp. $r^s$ and $r^l$. To assess results regarding the level, we run two regressions, one for each of the curve. We compute the slope $\Delta$ as the difference between the long yield and the short one. The debt-to-GDP ratio $d$ is a proxy for the bond volumes. As we explained before, the job-loss probability $q$ approximates the credit constraint probability. Since we consider a panel data, we denote $x_{it}$ the value of the variable $x$ for the country $i$ in year $t$. The set of countries (resp. years) is denoted $I$ (resp. $T$).

Naive estimation. The first step consists simply in regressing the level and the slope of the curve on the debt-to-GDP ratio and the job-loss probability:

\[
\begin{align*}
    x = s, l \\
    r^x_{it} &= \alpha^x.N + \beta^x.d_{it} + \beta^x.q_{it} \\
    \Delta_{it} &= \alpha^{\Delta,N} + \beta^{\Delta,d}_{it} + \beta^{\Delta,q}_{it}
\end{align*}
\]

Results in table (Tab. 1) are not very convincing. The coefficients are not always significant and do not always have the right sign. In particular, the impact of job-loss probability on the

\footnote{Countries are: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, Netherlands, Norway, Spain, Sweden, the United Kingdom, and the United States}
level of the curve is significantly positive, which is the opposite of our theoretical findings. However, the debt-to-GDP ratio is more in line with our predictions, except for the short rate, where it is negative but not significant.

We then add country and time fixed effects to control for both variables. We suppose that these fixed effects are additive and separable. The regression (34) becomes:

\[
\begin{align*}
    x = s, l, \\
    r_{it}^x = \alpha^{x,N} + \beta_d^{x,N} y_{it} + \beta_d^{x,N} q_{it} + \text{dummies}_{\text{country}} + \text{dummies}_{\text{year}} \\
    \Delta_{it} = \alpha^{\Delta,N} + \beta_d^{\Delta,N} y_{it} + \beta_d^{\Delta,N} q_{it} + \text{dummies}_{\text{country}} + \text{dummies}_{\text{year}}
\end{align*}
\]

To avoid unnecessarily complex notations, we gather the control effects under the term of \(\text{dummies}_{\text{country}}\) and \(\text{dummies}_{\text{year}}\). For example \(\text{dummies}_{\text{country}}\) is equal to the sum of 17 country indicator functions. The first result is that the contribution of the debt-to-GDP ratio is always positive and significant (except for the slope however): A larger bond volume seems therefore to modify the shape of the curve, as our model predicts. Nonetheless, the role of the job loss probability is much less in line with our findings. Even if the effect on the slope is negative (but not significant), the effect on the level (for both the short end and the long end) is significantly positive, which is the exact opposite of our model predictions.

**First regressions**

<table>
<thead>
<tr>
<th>Regression of the dependent variables</th>
<th>Without Fixed Effects</th>
<th>With Fixed Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Slope</td>
<td>Long Yield</td>
</tr>
<tr>
<td>Job-loss (q)</td>
<td>0.255**</td>
<td>0.414**</td>
</tr>
<tr>
<td>Std Err</td>
<td>0.115</td>
<td>0.205</td>
</tr>
<tr>
<td>Debt-to-GDP ratio</td>
<td>0.010***</td>
<td>0.001</td>
</tr>
<tr>
<td>Std Err</td>
<td>0.002</td>
<td>0.005</td>
</tr>
<tr>
<td>Country Fixed Effect</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Year Fixed Effect</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>(F)-stat</td>
<td>8.04</td>
<td>2.26</td>
</tr>
<tr>
<td>(R^2)</td>
<td>4.31%</td>
<td>1.07%</td>
</tr>
</tbody>
</table>

*** significant at 1%, ** at 5% and, * at 10%.

**Table 1. First regressions**

**Instrumentation.** We now instrument the job-loss probability in order to have a better estimator of the credit constraint probability. The main reason is that it seems reasonable to believe that this probability does not only reflect the idiosyncratic risk and the probability of facing credit constraints. First, the flow into unemployment include both voluntary resignations and dismissals. The correlation between GDP growth and the job-loss probability
is slightly positive (6% in our sample). Since resignations are more likely to occur in booms than dismissals, this positive correlation illustrates that the job-loss probability is not a perfect proxy for falling voluntarily into unemployment and thus for facing credit constraints. Second, this job-loss probability \( q \) may reflect labor market institutions and its flexibility. When computing this average probability across time by country, we obtain that it is notably very low (about .5% each month) in Continental Europe (Austria, Belgium, France, Germany, Italy, Netherlands, Switzerland) and in Japan. It is larger in Scandinavia (Denmark, Finland and Sweden) and again larger (btw. 1.5 and 2.5%) in Commonwealth countries: Australia, Canada, United Kingdom and United States. The correlation of the average job loss probability with the inverse of the Index of Strictness of Employment Protection Legislation (EPL) in 2003\(^{17}\) is roughly equal 50% (and larger than 70% if we exclude Spain, who seems to have a very rigid labor market according to EPL and to experience large job-loss probabilities).

The both preceding remarks offer a clue to instrumenting the job-loss probability \( q \).

### 8.1.3 Results

Before presenting the final results, we explicit the instrumentation. The regression results of the job-loss probability on the unemployment are gathered in the table (Tab. 2). We do not explicit the coefficients of the control dummies. We can make three remarks about these results. First, the unemployment coefficient is significantly (at the 1% level) negative. The larger the stock of unemployed people, the smaller the inflows into unemployment. Second, the debt-to-GDP ratio, which enters the regression as an exogenous variable, influences also in a significant and negative way the job-loss probability. Finally, the Fisher statistic is roughly 79: The rule of thumb stating that the F stat should be larger than 10 is thus valid. The R-square is also very large and reaches 79.9%.

The instrumentation does not suffer from overidentification, since we instrument a single variable (job-loss probability \( q \)), with a single instrument (unemployment). We additionally test the exogeneity of the instrumental variable. The idea consists in checking whether the regression results with MCO and with the instrumentation are statically different. We use an augmented regression of the dependent variable (short or long yield or slope) on the exogenous variables, meaning the debt-to-GDP ratio and the dummies, and on the job-loss probability, as

\(^{17}\)The OCDE computes this indicator as a proxy of labor market flexibility. It includes notably rules on dismissals, the use of fixed and temporary contracts. Values exist for in 1990, 1998 and 2003. The index is high for flexible markets, so that we compute the inverse, to obtain a positive correlation with the job-loss probability.
**Instrument regression**

*Regression of the job-loss probability $q$*

<table>
<thead>
<tr>
<th>$q$</th>
<th>Without control</th>
<th>Control</th>
<th>Real GDP Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment ($\times 10^{-3}$)</td>
<td>-0.236***</td>
<td>-0.226***</td>
<td></td>
</tr>
<tr>
<td>(Std Err) ($\times 10^{-3}$)</td>
<td>(0.089)</td>
<td>(0.089)</td>
<td></td>
</tr>
<tr>
<td>Debt-to-GDP ratio ($\times 10^{-3}$)</td>
<td>3.121***</td>
<td>2.643***</td>
<td></td>
</tr>
<tr>
<td>(Std Err) ($\times 10^{-3}$)</td>
<td>(0.983)</td>
<td>(1.085)</td>
<td></td>
</tr>
<tr>
<td>Real GDP Growth ($\times 10^{-3}$)</td>
<td>-</td>
<td>-22.268</td>
<td></td>
</tr>
<tr>
<td>(Std Err) ($\times 10^{-3}$)</td>
<td>-</td>
<td>(16.480)</td>
<td></td>
</tr>
<tr>
<td>Country Fixed Effect</td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>Year Fixed Effect</td>
<td>YES</td>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>F-stat</td>
<td>79.02</td>
<td>70.29</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>79.90%</td>
<td>80.14%</td>
<td></td>
</tr>
</tbody>
</table>

*** significant at 1%, ** at 5 % and, * at 10%.

**Table 2. Instrumentation**

We estimate the equality between MCO and IV estimators, if the coefficient of the estimated job-loss probability is significantly different from 0. As the table (Tab. 3) shows, the test is valid (except for the short rate), meaning that IV and MCO estimators are different.

**Exogeneity test**

*Regression of the dependent variables*

<table>
<thead>
<tr>
<th>Job-loss $q$</th>
<th>Without Control</th>
<th>Control with GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>0.736</td>
<td>0.482**</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>(0.204)</td>
<td>(0.214)</td>
</tr>
<tr>
<td>Estim. Job-Loss $\hat{q}$</td>
<td>$-3.954^*$</td>
<td>$-3.954^{**}$</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>(2.230)</td>
<td>(2.100)</td>
</tr>
<tr>
<td>Debt-to-GDP ratio</td>
<td>0.014**</td>
<td>0.400***</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>(0.007)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>Real GDP Growth</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Country Fixed Effect</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>Year Fixed Effect</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>F-stat</td>
<td>13.28</td>
<td>28.63</td>
</tr>
<tr>
<td>$R^2$</td>
<td>54.46%</td>
<td>78.51%</td>
</tr>
</tbody>
</table>

*** significant at 1%, ** at 5 % and, * at 10%.

**Table 3. Exogeneity Regression**

The table (Tab. 4) gathers the results of the second stage regression, for the three dependent variables: the real short rate, the real long rate, and the slope. Conclusions are twofold. First, the debt-to-GDP ratio still influences positively and significantly all variables. Second, the job-
loss probability has a significantly negative impact on the long rate and the slope. It is however highly insignificant for the short rate (p-value larger than 90%). This result therefore confirms to a large extent our theoretical findings: The tighter the credit constraints, the smaller the yields and the slope.

### Results

<table>
<thead>
<tr>
<th>Regression of the dependent variables</th>
<th>Without Control</th>
<th>Control with GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Slope</td>
<td>Long Yield</td>
</tr>
<tr>
<td>Estim. Job-Loss $\hat{q}$</td>
<td>$-3.740^*$</td>
<td>$-3.468^*$</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>(2.227)</td>
<td>(2.093)</td>
</tr>
<tr>
<td>Debt-to-GDP ratio</td>
<td>0.014**</td>
<td>0.400***</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>(0.007)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>GDP growth</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>(Std Err)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Country Fixed Effect</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>Year Fixed Effect</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>F-stat</td>
<td>13.49</td>
<td>29.32</td>
</tr>
<tr>
<td>$R^2$</td>
<td>54.44%</td>
<td>78.21%</td>
</tr>
</tbody>
</table>

*** significant at 1%, ** at 5 % and, * at 10%.

Table 4. IV Results

#### 8.1.4 Robustness Checks

We run two kinds of robustness checks: (i) we add the real GDP growth as a control variable, (ii) we modify the proxy for the real rate: We replace the nominal deflated by the expected inflation with the nominal rate deflated by the GDP deflator.

We have not reported the results of naive regressions in table (Tab. 5) below, but other ones (first stage regression, exogeneity test and final regression) are reported respectively in tables (Tab. 2), (Tab. 3), and (Tab. 4).

The addition of the GDP growth does not change results in a significant way. However, we can remark one interesting point. If the coefficient for the slope regression is not significant, it is significantly negative in both level regressions. As stated in the lemma (1) (even if this result does not strictly rely on credit constraints), the yield curve in the bad state of the world lies above the one in the good state.

#### 8.1.5 Empirical conclusions

After preceding results, we can draw three conclusions:
First regressions with control variable

Regression of the dependent variables

<table>
<thead>
<tr>
<th></th>
<th>Without Fixed Effects</th>
<th>With Fixed Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Slope</td>
<td>Long Yield</td>
</tr>
<tr>
<td>Job-loss ( q )</td>
<td>0.276**</td>
<td>0.398**</td>
</tr>
<tr>
<td>Std Err</td>
<td>0.108</td>
<td>0.202</td>
</tr>
<tr>
<td>Debt-to-GDP ratio</td>
<td>0.011***</td>
<td>-0.001</td>
</tr>
<tr>
<td>Std Err</td>
<td>0.002</td>
<td>0.005</td>
</tr>
<tr>
<td>Real GDP Growth</td>
<td>0.115***</td>
<td>-0.086</td>
</tr>
<tr>
<td>Std Err</td>
<td>0.039</td>
<td>0.069</td>
</tr>
<tr>
<td>Country Fixed Effect</td>
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<td>NO</td>
</tr>
<tr>
<td>Year Fixed Effect</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>F-stat</td>
<td>9.16</td>
<td>1.97</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>6.97%</td>
<td>1.45%</td>
</tr>
</tbody>
</table>

*∗∗∗ significant at 1%, ∗∗ at 5% and, ∗ at 10%.

Table 5. First regressions with control variable

(i) The volume effect is robust and confirms our theoretical findings. It appears as soon as we account for fixed effects.

(ii) The credit constraint effect measurement suffers from the quality of the proxy, but a reasonable instrumentation allows us to identify a significant effect in line with our results.

(iii) The control by the real GDP growth does not modify previous results. They confirm our ranking of yield curves across states of the world.

9 Concluding remarks

This paper provides a tractable yield curve equilibrium model in an incomplete market economy, where agents face both idiosyncratic and aggregate shocks. We derive analytical expressions for bond prices. They allow us to characterize, how government debt issuances and credit constraints may affect the shape of the average yield curve. Our model predicts that a larger bond volume raises both its level and its slope, whereas credit constraints have an opposite influence. We also prove that the market incompleteness contributes to the rejections of the expectations hypothesis. Using a panel regression, we have proved that empirical data in OECD countries validate our theoretical findings, at least regarding the impact of bond volumes and credit constraints on the level and the slope of the curve.

A straightforward route for future research is to extend the model to confront it to the data and to measure to what extent market incompleteness affects interest rate dynamics.
Three simple extensions can be easily introduced. The first one is to make stochastic processes more realistic. The number of aggregate states could easily be increased to reproduce more closely macroeconomic time series dynamics, such as the labor income one. Second, in our model, we assumed for simplification purposes that the job-loss probabilities are constant. The correlation between this probability and the aggregate shock could be easily introduced as in Mankiw (1986) or Krusell and Smith (1997). Changes in the probability of being unemployed and thus of being credit-constrained will moreover distort the yield curve: Level, and slope are thus likely to be modified. Third and more importantly, this model suggests that the dynamics of bond volumes, used as liquidity devices, should be explicitly introduced in the dynamics of the yield curve.
References


Appendices

A  Proof of proposition 1

We prove that $C_s^k$ are $C^1$ functions of $B_i$ and $z^h, z^l$ for $s = h, l$ and $k, i = 1, \ldots, n$. We define the following matrices ($u^h$ and $u^l$ are defined in (25)):

$$C = [C_n^h \ C_1^h \ \ldots \ C_1^h \ C_0^h \ C_0^l]^T$$
$$X = [z^h \ z^l \ B^T] \text{ with } B = [B_n \ \ldots \ B_1]^T$$

$$M(C, X) = \beta \left[ \begin{array}{ccc} \pi^h (\alpha + (1 - \alpha) z^h u^h) & (1 - \pi^h) (\alpha + (1 - \alpha) z^h u^l) \\ (1 - \pi^l) (\alpha + (1 - \alpha) z^l u^h) & \pi^l (\alpha + (1 - \alpha) z^l u^l) \end{array} \right] \quad (36)$$

The price structure (26) expresses using preceding notations as ($0_{m \times n}$ is the $m \times n$ null matrix; if $m = 0$ or $n = 0$, then the matrix has no dimension):

$$f(C, X) \equiv C - \begin{bmatrix} 0_{2 \times 2} & M(C, X) & 0_{2 \times 2} & \ldots & 0_{2 \times 2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & M(C, X) & \ddots & \vdots \\
0_{2 \times 2} & \ldots & 0_{2 \times 2} & \ddots & 0_{2 \times 1} \end{bmatrix} C - \begin{bmatrix} 0 \\
\vdots \\
0 \\
1/z^h \\
1/z^l \end{bmatrix} = 0_{(2n+2) \times 1}$$

Since $u'$ is $C^1$ on $\mathbb{R}$, $M$ and $f$ are also $C^1$ in $(C, X)$. Before using the implicit function theorem to show that $C$ is $C^1$ in $(B, X)$, we prove that the Jacobian $Df_y = (\frac{\partial f}{\partial C_1^h}, \frac{\partial f}{\partial C_1^n}, \ldots, \frac{\partial f}{\partial C_{n-i}^n})$ of $f$ relative to $C$ is invertible.

As in (25) for $u^h$ and $u^l$, we define $u^{nh}$ and $u^{nl}$ as follows:

$$u^{nh} \equiv u'' \left( \delta + \sum_{i=0}^{n-1} C_i^h z^h B_{i+1}/\omega^e \right) \quad \text{and} \quad u^{nl} \equiv u'' \left( \delta + \sum_{i=0}^{n-1} C_i^l z^l B_{i+1}/\omega^e \right) \quad (37)$$

We express partial derivatives of $f$ relative to $C_{n-i}^n$ ($i = 0, \ldots, n-1$).

First, derivatives relative to $C_n^h$ and $C_1^n$ are:

$$\frac{\partial f}{\partial C_n^h} = \begin{bmatrix} 1 \\
0 \\
0_{2n \times 1} \end{bmatrix} \quad \text{and} \quad \frac{\partial f}{\partial C_l^n} = \begin{bmatrix} 0 \\
1 \\
0_{2n \times 1} \end{bmatrix}$$
Second, derivatives relative to $C_{n-i}^h$ and $C_{n-i}^l$ for $1 \leq i \leq n$ are:

$$\frac{\partial f}{\partial C_{n-i}^h} = \begin{bmatrix}
0_{2(i-1) \times 1} \\
-\beta \left( (1-\alpha)z^h u^h \right) \pi^h \\
-\beta \left( (1-\alpha)z^h u^h \right) (1-\pi^l) \\
1 \\
0 \\
0_{2(n-i) \times 1}
\end{bmatrix}$$

$$\frac{\partial f}{\partial C_{n-i}^l} = \begin{bmatrix}
0_{2(i-1) \times 1} \\
-\beta \left( (1-\alpha)z^l u^l \right) (1-\pi^h) \\
-\beta \left( (1-\alpha)z^l u^l \right) \pi^l \\
0 \\
1 \\
0_{2(n-i) \times 1}
\end{bmatrix}$$

The Jacobian $Df_Y$ expresses as the sum of an upper triangular matrix with only 1 on its diagonal and a matrix which is equal to 0 when $B = 0$. Close to the zero net supply, the Jacobian is invertible and $C$ is a $C^1$ function of $B$ and of $\{z^h, z^l\}$. QED.

### B Proof of lemma 1

#### B.1 Ranking of yield curves.

We prove by inference in the zero net supply case ($B = 0$) that $C_{n-i}^h z^h > C_{n-i}^l z^l$ for $k = 1, \ldots, n$ if $\pi^h + \pi^l > 1$ using price definitions (26).

1. The result holds for $k = 1$. Indeed, substituting $C_1^h$ and $C_1^l$ using (26), one finds that $C_1^h z^h > C_1^l z^l$ is equivalent to

$$\alpha \left( \pi^h - (1-\pi^l) z^l_{\pi^h} + \frac{z^h}{z^l} \left( (1-\pi^h) - \pi^l \right) \right) > (1-\alpha) u'(\delta) \left( \pi^l z^l + (1-\pi^l) z^l - \pi^h z^h - (1-\pi^h) z^h \right)$$

Since $z^h \geq z^l$ and $\pi^h + \pi^l - 1 > 0$, one can check that the left hand side is strictly positive whereas the right hand side is strictly negative.
2. For a given maturity $k \geq 2$, let us suppose that the result holds for the previous maturity:

$$C_{k-1}^h z^h > C_{k-1}^l z^l.$$ Proving our result $C_k^h z^h > C_k^l z^l$ is equivalent to:

$$\alpha \left( \left( \pi^h - (1 - \pi^l) \frac{z^l}{z^h} \right) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l \right) \geq (1 - \alpha) u'(\delta) \left( \frac{z^l}{z^h} + (1 - \pi^l) z^l - \pi^h z^h \right) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} - (1 - \pi^h) z^h \right) \tag{39}$$

First, consider the right hand side $RHS$. Since $z^h \geq z^l$ and $\pi^h + \pi^l - 1 > 0$, we have $(1 - \pi^l) z^l - \pi^h z^h < 0$. By assumption $\frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} \geq 1$, the $RHS$ thus verifies:

$$RHS < (1 - \alpha) u'(\delta) \left( \pi^l z^l + (1 - \pi^l) z^l - \pi^h z^h - (1 - \pi^h) z^h \right) < (1 - \alpha) u'(\delta) (z^l - z^h) < 0$$

Second, consider the LHS. Using the same argument as for the RHS, we obtain:

$$\text{LHS} > \alpha \left( \pi^h - \pi^l - (1 - \pi^l) \frac{z^l}{z^h} + \frac{z^h}{z^l} (1 - \pi^h) \right) > \alpha \left( z^h - z^l \right) \left( \frac{1 - \pi^h}{z^l} + \frac{1 - \pi^l}{z^h} \right) > 0$$

By inference, we obtain $C_k^h z^h > C_k^l z^l$ for $k = 1, \ldots, n$ if $\pi^h + \pi^l > 1$.

The result is true in zero supply and still holds, by continuity, for small bond volumes. $QED$.

### B.2 Value of the long run interest rate

We determine the common value, towards which yields converge in both states. We diagonalize the matrix $M(C, X)$ defined in (36) as $M(C, X) = \beta Q D Q^{-1}$. Matrices $Q$ and $D$ are defined as:

$$Q \equiv \begin{bmatrix} 1 & -\frac{(\pi^h - \pi^l) - (1 - \alpha)(\pi^h + \pi^l - \alpha(\pi^h + \pi^l)) - H}{2(1 - \pi^l)(\pi^h + \alpha(\pi^h + \pi^l))} \\ 1 & -\frac{(\pi^h - \pi^l) + (1 - \alpha)(\pi^h + \pi^l - \alpha(\pi^h + \pi^l)) + H}{2(1 - \pi^l)(\pi^h + \alpha(\pi^h + \pi^l))} \end{bmatrix}$$

$$D \equiv \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$H \equiv \begin{bmatrix} \alpha \left( \pi^h + \pi^l \right) + (1 - \alpha) \left( z^h u^h \pi^h + z^l u^l \pi^l \right) - H \\ 0 & \alpha \left( \pi^h + \pi^l \right) + (1 - \alpha) \left( z^h u^h \pi^h + z^l u^l \pi^l \right) + H \end{bmatrix}$$

(One can check that the term under the square is always positive in the neighborhood of the zero net supply). We can now diagonalize the matrix defining $C^h$ and $C^l$ and modify the system (26) and iterate it to obtain:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \beta^k Q D^k Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix} \implies \begin{bmatrix} P_k^h \\ P_k^l \end{bmatrix} = \beta^k \begin{bmatrix} z^h & 0 \\ 0 & z^l \end{bmatrix} Q D^k Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix}$$
Developing the preceding equality allows us to obtain an analytical expression for $P^h_k$. Re-marking that $H > 0$, we simplify the expression with $\left(\frac{\alpha\left(\pi^h + \pi^l\right) - (\alpha - 1)(z^hu^h\pi^h + z^lu^l\pi^l) - H}{\alpha\left(\pi^h + \pi^l\right) - (\alpha - 1)(z^hu^h\pi^h + z^lu^l\pi^l) + H}\right)^k \rightarrow 0$. As a consequence, the price $P^h_k$ verifies:

\[
\lim_{k \to \infty} \log \left[ P^h_k \left( \frac{1}{H^2} \right)^{2-k-1} \beta^k \right] = \log \left[ H - \left( \alpha\left(\pi^l - \pi^h\right) + (\alpha - 1)(z^hu^h\pi^h - z^lu^l\pi^l) + 2(\alpha + z^lu^l)(1 - \alpha)(1 - \pi^h) \right) \right]
\]

As $r^h_k = -\frac{1}{k} \log P^h_k$, one finally deduces the expression of $r^h_k$ when its maturity goes to infinity. By a simple symmetry argument, one obtains the same expression for the common limit:

\[
\lim_{k \to \infty} r^h_k = \lim_{k \to \infty} r^l_k = r^\text{lim} = -\log \beta - \log \left( \frac{\alpha\left(\pi^h + \pi^l\right) + (1 - \alpha)(z^hu^h\pi^h + z^lu^l\pi^l) + H}{2} \right)
\]

C Equilibrium of the simple model

For this equilibrium to exist, unemployed must be credit constrained in all states of the world. Using directly general conditions (29), we obtain for all $t > 0$:

\[
C_1 z_t u' \left( \delta + B_1 + C_1 z_t B_2 \right) > \beta \left( 1 - \rho \right) \phi E_t 1/z + \beta \rho u' \left( \delta \right)
\]

\[
C_2 z_t u' \left( \delta + B_1 + C_1 z_t B_2 \right) > \beta \left( 1 - \rho \right) \phi C_1 + \beta \rho u' \left( \delta \right) C_1
\]

The sufficient condition for the equilibrium existence expresses as:

\[
\max \left( \frac{C_2}{C_1}, C_1 \right) \left( u_1 - u_2 \left( \delta + B_1 + C_1 z^h B_2 \right) \right) > \frac{\beta}{z} \left( (1 - \rho) \phi E_t \left[ \frac{1}{z} \right] + \rho \left( u_1 - u_2 \delta \right) \right)
\]

It is noteworthy that this condition implies that $u$ fulfills conditions of monotony and the concavity. In order to verify that $u$ is strictly increasing, we only need to check that $u_1 - u_2 \left( \delta + B_1 + C_1 z^h B_2 \right)$, which is implicitly verified in the preceding condition. The concavity means only that $u_2 > 0$. 

47
D Proof of proposition 2

D.1 Impact of credit constraints on prices

We prove by inference that credit constraints decrease yields. We prove the result for \( C_h \) and the method is the same for \( C_l \). We begin with expressing the derivative of \( C_h \) relative to \( \alpha \) (\( u^{\prime\prime} \) and \( u^{\prime\prime\prime} \) are defined in (37)):

\[
\frac{\partial C_h}{\partial \alpha} = \alpha \beta \left( \pi^h \frac{\partial C_{k-1}^h}{\partial \alpha} + (1 - \pi^h) \frac{\partial C_{k-1}^l}{\partial \alpha} \right) + (1 - \alpha) \beta \left( \pi^h \frac{\partial C_{k-1}^h}{\partial \alpha} z^h u^h + (1 - \pi^h) \frac{\partial C_{k-1}^l}{\partial \alpha} z^l u^l \right) \\
+ (1 - \alpha) \beta \pi^h C_{k-1}^h (z^h)^2 \sum_{j=1}^{n} \frac{\partial C_{j-1}^h}{\partial \alpha} B_j u^{\prime\prime} + (1 - \alpha) \beta (1 - \pi^h) C_{k-1}^l (z^l)^2 \sum_{j=1}^{n} \frac{\partial C_{j-1}^l}{\partial \alpha} B_j 
\]

(41)\[(42)\]

1. The result holds for \( k = 1 \), since the derivative (42) provides for small bond supply (assumption B):

\[
\frac{\partial C_1^h}{\partial B_i} \approx \beta \pi^h / z^h (1 - z^h u'(\delta)) < 0
\]

2. We suppose that the result holds for \( k - 1 \) and \( \frac{\partial C_{k-1}^h}{\partial \alpha}, \frac{\partial C_{k-1}^l}{\partial \alpha} < 0 \). \( \frac{\partial C_{k-1}^l}{\partial \alpha} B_j \) is negligible, the Eq. (42) implies that \( \frac{\partial C_k^h}{\partial \alpha} < 0 \). Larger credit constraints (i.e. smaller \( \alpha \)) increases prices. Q.E.D.

D.2 Impact of credit constraints on the slope

We prove that credit constraints contribute to flatten the curve. Using the expression (40) of \( \tilde{r}^{\text{lim}} \) and the one of \( r_1 \) ((27) and the price structure (26)), the derivative of the slope relative to \( \alpha \) expresses as:

\[
\frac{\partial}{\partial \alpha} \Delta = \frac{(1 - \pi^h)(1 - \pi^l)(z^h - z^l)^2 (\pi^l z^h + \pi^h z^l) u'(\delta)}{(2 - \pi^h - \pi^l) (\pi^l z^h + (1 - \pi^l) z^l) ((1 - \pi^h) z^h + \pi^h z^l)} + O(1 - \alpha)
\]

Without ambiguity, \( \frac{\partial}{\partial \alpha} \Delta > 0 \) in zero net supply and for small credit constraints. By continuity, the result still holds for small bond volumes: Credit constraints flatten the curve. Q.E.D.
D.3 Impact of credit constraints on the variance

We prove by inference that the variance of yields of any maturity increases with credit constraints. Showing that the variance increases with credit constraints is equivalent to:

$$\frac{\partial \text{Var}(r_j)}{\partial \alpha} < 0 \iff (r_j^h - r_j^l) \frac{\partial (r_j^h - r_j^l)}{\partial \alpha} < 0 \iff \forall k \geq 1, \forall j \geq 1, \quad \mathcal{V}_j = \frac{1}{C_j^l} \frac{\partial C_j^l}{\partial \alpha} - \frac{1}{C_j^h} \frac{\partial C_j^h}{\partial \alpha} > 0$$

We compute $\mathcal{V}_j$ ($j \leq 1$), which measures the impact of credit constraints on the variance:

$$\frac{\mathcal{V}_j}{\beta} = \left( \frac{\pi^h}{C_j^h} - \frac{1 - \pi^l}{C_j^l} \right) C_{j-1}^h \left( 1 - z^h u'(\delta) \right) - \left( \frac{\pi^l}{C_j^l} - \frac{1 - \pi^h}{C_j^h} \right) C_{j-1}^l \left( 1 - z^l u'(\delta) \right)$$

$$+ \left( \frac{\pi^h}{C_j^h} - \frac{1 - \pi^l}{C_j^l} \right) C_{j-1}^h \left( \alpha + (1 - \alpha) z^h u'(\delta) \right) \frac{1}{C_{j-1}^h} \frac{\partial C_{j-1}^h}{\partial \alpha}$$

$$- \left( \frac{\pi^l}{C_j^l} - \frac{1 - \pi^h}{C_j^h} \right) C_{j-1}^l \left( \alpha + (1 - \alpha) z^l u'(\delta) \right) \frac{1}{C_{j-1}^l} \frac{\partial C_{j-1}^l}{\partial \alpha}$$

1. For $j = 1$, the expression of $\mathcal{V}_1$ simplifies to:

$$\frac{\mathcal{V}_1}{\beta} = \frac{(\pi^h + \pi^l - 1) z^h z^l (z^h - z^l) u'(\delta)}{C_1^l C_1^h} > 0$$

2. We suppose that the result holds for a given $j - 1$. In order to determine the sign of $\mathcal{V}_j$, we proceed in two steps and consider: (i) the first two terms and (ii) the last two ones of the expression (43). First, remark that:

(i) $$\left( \frac{\pi^h}{C_j^h} - \frac{1 - \pi^l}{C_j^l} \right) \left( 1 - z^h u'(\delta) \right) C_{j-1}^h - \left( \frac{\pi^l}{C_j^l} - \frac{1 - \pi^h}{C_j^h} \right) \left( 1 - z^l u'(\delta) \right) C_{j-1}^l =$$

$$- \beta C_{j-1}^l C_{j-1}^h (\pi^h + \pi^l - 1)(z^h - z^l) u'(\delta) \frac{\partial C_{j-1}^h}{\partial \alpha} < 0$$

(ii) $$\left( \frac{\pi^h}{C_j^h} - \frac{1 - \pi^l}{C_j^l} \right) C_{j-1}^h \left( \alpha + (1 - \alpha) z^h u'(\delta) \right) = \left( \frac{\pi^l}{C_j^l} - \frac{1 - \pi^h}{C_j^h} \right) C_{j-1}^l \left( \alpha + (1 - \alpha) z^l u'(\delta) \right)$$

$$= \beta (\pi^h + \pi^l - 1) C_{j-1}^l C_{j-1}^h \left( \alpha + (1 - \alpha) z^l u'(\delta) \right) \left( \alpha + (1 - \alpha) z^h u'(\delta) \right) > 0$$

Using the inference hypothesis $\frac{1}{C_{j-1}^l} \frac{\partial C_{j-1}^h}{\partial \alpha} < \frac{1}{C_{j-1}^h} \frac{\partial C_{j-1}^l}{\partial \alpha}$, we obtain that the last two terms are negative. We conclude that $\mathcal{V}_j > 0$.

By continuity, the result holds, when bond volumes are not too large. \textit{QED.}
E Proof of proposition 3

E.1 Impact of bond volumes on prices

We prove by inference that bond volumes decrease prices and increase yields. We prove the result for $C_h^k$ and the method is the same for $C_l^k$. We begin with expressing the derivative of $C_h^k$ relative to $B_i$ for $1 \leq k, i \leq n$ ($u''$ and $u'''$ are defined in (37)):

$$\frac{\partial C_h^k}{\partial B_i} = \alpha \beta \left( \pi^h \frac{\partial C_h^{k-1}}{\partial B_i} + (1 - \pi^h) \frac{\partial C_l^{k-1}}{\partial B_i} \right) + (1 - \alpha) \beta \left( \pi^h \frac{\partial C_h^{k-1}}{\partial B_i} z^h u'' + (1 - \pi^h) \frac{\partial C_l^{k-1}}{\partial B_i} z^l u'' \right)$$

$$+ (1 - \alpha) \beta \pi^h C_h^{k-1} (z^h)^2 \left( \sum_{j=1}^{n} \frac{\partial C_h^{j-1}}{\partial B_i} B_j + C_{i-1}^h \right) u''$$

$$+ (1 - \alpha) \beta (1 - \pi^h) C_{k-1}^l (z^l)^2 \left( \sum_{j=1}^{n} \frac{\partial C_l^{j-1}}{\partial B_i} B_j + C_{i-1}^l \right) u'''$$

(44)

1. The result holds for $k = 1$, since the derivative (44) provides for small bond supply:

$$\frac{\partial C_h^1}{\partial B_i} \approx (1 - \alpha) \beta \left[ \pi^h C_h^{0} (z^h)^2 C_{i-1}^h u'' + (1 - \pi^h) C_{k-1}^l (z^l)^2 C_{i-1}^l u''' \right]$$

2. We suppose that the result holds for $k - 1$ and $\frac{\partial C_h^{k-1}}{\partial B_i}, \frac{\partial C_l^{k-1}}{\partial B_i} < 0$. Since $C_s^s$ is a $C^1$ function of $B_i$, $\frac{\partial C_s^s}{\partial B_i}$ is continuous in $B_i$ and $B_j \frac{\partial C_s^s}{\partial B_i}$ (s = h, l) is negligible relative to $C_{i-1}^s$ for small bond supply. Eq. (44) implies that $\frac{\partial C_h^k}{\partial B_i} < 0$. Larger bond supply decreases prices. QED.

E.2 Impact of bond volumes on the slope

We prove that larger bond volumes steepen the curve. Using the expression (40) of $\pi_{lim}$ and the one of $r_1$ ((27) and the price structure (26)), the derivative of the slope relative to $B_j$ for $j \leq n$ expresses as:

$$\frac{\partial}{\partial B_j} \Delta = (1 - \alpha) \frac{(1 - \pi^h)(1 - \pi^l)(z^h - z^l)}{2 - \pi^h - \pi^l} \times \left( \frac{z^l}{\pi^l z^h + (1 - \pi^l)z^l} \frac{\partial u''}{\partial B_j} - \frac{z^h}{(1 - \pi^h)z^h + \pi^h z^l} \frac{\partial u'''}{\partial B_j} \right) + O \left( (1 - \alpha)^2 \right)$$

50
with (we also give expressions when the bond supply is close to 0):

\[
\frac{\partial}{\partial B_j} u'' = z^h \left( \sum_{k=1}^{n} \frac{\partial C^h_k}{\partial B_j} B_{k+1} + C^h_{j+1} \right) u'' \left( \delta + B_1 + z^h \sum_{k=1}^{n} C^h_k B_{k+1} \right) \approx z^h C^h_{j-1} u''(\delta)
\]

\[
\frac{\partial}{\partial B_j} u'' = z^l \left( \sum_{k=1}^{n} \frac{\partial C^l_k}{\partial B_j} B_{k+1} + C^l_{j+1} \right) u'' \left( \delta + B_1 + z^l \sum_{k=1}^{n} C^l_k B_{k+1} \right) \approx z^l C^l_{j-1} u''(\delta)
\]

The sign of \(\frac{\partial}{\partial B_j} \Delta\) depends on the sign of \(A = \frac{z^l}{\pi z^n + (1-\pi)z^l} \frac{\partial C^l_k}{\partial B_j} - \frac{z^h}{\pi z^n + (1-\pi)z^h} \frac{\partial C^h_k}{\partial B_j}\). Substituting the derivatives by their values allows to express \(A\) when bond supply is close to 0 as:

\[
\tilde{A} = \left( \frac{z^l}{\pi (z^h - z^l) + z^l C^l_{j-1} z^l - \frac{z^h}{\pi (z^h - z^h) + z^h C^h_{j-1} z^h} \right) u''(\delta)
\]

As \(z^h > z^l\), a sufficient condition for \(\tilde{A}\) to be positive is \(C^h_{j-1} z^h > C^l_{j-1} z^l\), which is always true. As a conclusion, bond volumes increase the slope of the curve as soon as \(\pi^h + \pi^l > 1\). QED.

### E.3 Impact of bond volumes on the variance

We prove by inference that the variance of yields of any maturity decreases with the supply of bonds, whatever its maturity. Showing that the variance decreases is equivalent to:

\[
\frac{\partial \text{Var}(r_j)}{\partial B_k} < 0 \iff (r^h_j - r^l_j) \frac{\partial (r^h_j - r^l_j)}{\partial B_k} \leq 0 \iff \forall k \geq 1, \forall j \geq 1, \frac{1}{C^l_j} \frac{\partial C^l_j}{\partial B_k} - \frac{1}{C^h_j} \frac{\partial C^h_j}{\partial B_k} > 0
\]

Price definitions are recalled in the appendix in the proof regarding the impact of credit constraints.

The result is true for \(j = 0\). We suppose now the result true for maturity \(j - 1\) and we prove now the result for a given price of maturity \(j\). The derivative of \(C^h_j\) relative to \(B_{k+1}\) expresses as:

\[
\frac{1}{\beta} \frac{\partial C^h_j}{\partial B_{k+1}} = \left( \frac{\alpha + (1-\alpha)z^h u'^h}{\pi^h} \frac{\partial C^h_{j-1}}{\partial B_{k+1}} + \frac{\alpha + (1-\alpha)z^l u'^l}{1-\pi^h} \left( 1 - \frac{\partial C^l_{j-1}}{\partial B_{k+1}} \right) \right) u''^h
\]

\[
+ \left( 1 - \alpha \right) \pi^h C^h_{j-1} \left( z^h \right)^2 \left[ \sum_{i=1}^{j} B_i \frac{\partial C^h_{i-1}}{\partial B_{k+1}} + C^h_k \right] u''^h
\]

\[
+ \left( 1 - \alpha \right) \left( 1 - \pi^h \right) C^l_{j-1} \left( z^l \right)^2 \left[ \sum_{i=1}^{j} B_i \frac{\partial C^l_{i-1}}{\partial B_{k+1}} + C^l_k \right] u''^l
\]

For small amounts of debt, expressions of derivatives of \(C^h_j\) and \(C^l_j\) simplify and the impact
of bond volumes on the variance depends on the sign of the following expression:

\[
\frac{1}{\beta} \frac{1}{C_j^h} \frac{\partial C_j^h}{\partial B_{k+1}} - \frac{1}{\beta} \frac{1}{C_j^l} \frac{\partial C_j^l}{\partial B_{k+1}} = \frac{1}{C_{j-1}^h(B_{k+1})} \frac{\partial C_{j-1}^h}{\partial B_{k+1}} \left( \alpha + (1 - \alpha) z^h u'(\delta) \right) \left( \pi^h C_{j-1}^h - (1 - \pi^h) C_{j-1}^l \right) + \frac{1}{C_j^l} \frac{\partial C_j^l}{\partial B_{k+1}} \left( \alpha + (1 - \alpha) z^l u'(\delta) \right) \left( (1 - \pi^h) C_{j-1}^l - \pi^l C_{j-1}^l \right) + (1 - \alpha) u''(\delta) \left[ z^h C_k^h z^h C_{j-1} \left[ \pi^h \frac{C_j^h}{C_j^l} - \frac{1 - \pi^h}{C_j^l} \right] - z^l C_k^l z^l C_{j-1} \left[ \pi^l \frac{C_j^l}{C_j^h} - \frac{1 - \pi^h}{C_j^h} \right] \right]
\]

We investigate this sign in two steps: (i) the last two terms and (ii) the first two ones. Let us begin with the last two terms and \( z^h C_{j-1}^h \left[ \pi^h \frac{C_j^h}{C_j^l} - \frac{1 - \pi^h}{C_j^l} \right] - z^l C_{j-1}^l \left[ \pi^l \frac{C_j^l}{C_j^h} - \frac{1 - \pi^h}{C_j^h} \right] \). The expression simplifies into:

\[
\frac{\alpha C_{j-1}^h C_{j-1}^l (\pi^h + \pi^l - 1) z^h z^l (z^h - z^l)}{C_{j-1}^l (1 - \pi^h) (\alpha + (1 - \alpha) u'(\delta) z^l) + C_{j-1}^h \pi^h (\alpha + (1 - \alpha) u'(\delta) z^h)} \times \frac{1}{C_{j-1}^l \pi^l (\alpha + (1 - \alpha) u'(\delta) z^l) + C_{j-1}^h (1 - \pi^h) (\alpha + (1 - \alpha) u'(\delta) z^h)}
\]

Because \( z^h C_{k}^h > z^l C_{k}^l \), the two last terms verify:

\[
(1 - \alpha) u''(\delta) \left[ z^h C_{j-1}^h z^h C_{j}^h \left[ \pi^h \frac{C_{j-1}^h}{C_j^l} - \frac{1 - \pi^l}{C_j^l} \right] - z^l C_{j-1}^l z^l C_{j}^l \left[ \pi^l \frac{C_j^l}{C_j^h} - \frac{1 - \pi^h}{C_j^h} \right] \right] < 0
\]

We consider now the two first terms. First remark the following equality:

\[
[\alpha + (1 - \alpha) z^h u'(\delta)] \left[ \pi^h \frac{C_{j-1}^h}{C_j^l} - (1 - \pi^l) \frac{C_{j-1}^l}{C_j^l} \right] = [\alpha + (1 - \alpha) z^l u'(\delta)] \left[ (1 - \pi^h) \frac{C_{j-1}^l}{C_j^l} - \pi^l \frac{C_{j-1}^l}{C_j^l} \right]
\]

\[
= \frac{1}{C_{j-1}^l} \frac{\partial C_{j-1}^h}{\partial B_{k+1}} \left( \alpha + (1 - \alpha) z^h u'(\delta) \right) (\alpha + (1 - \alpha) z^h u'(\delta)) > 0
\]

Using the inference condition \( \frac{1}{C_{j-1}^l} \frac{\partial C_{j-1}^h}{\partial B_{k+1}} < \frac{1}{C_j^l} \frac{\partial C_j^h}{\partial B_{k+1}} \), we conclude that the first two terms are also positive. As a conclusion, we obtain \( \frac{1}{C_j^l} \frac{\partial C_j^h}{\partial B_{k+1}} < \frac{1}{C_j^l} \frac{\partial C_j^l}{\partial B_{k+1}} \). The variance of all yields decreases with bond supply. \textit{QED.}

\section*{F Proof of Proposition 4}

We need a first order approximation in \( z^h - z^l \) of the regression coefficient \( \beta_n \).
F.1 A lemma

Lemma 2 The first order approximation for \( \beta_n \):

\[
\beta_n \approx 1 + \frac{(z^h - z^l)(\pi^h - \pi^l)}{2z^h(1 - t) t(\pi + (1 - \pi) (u'(\delta)/\phi) z^h)} \times \frac{n(t - t^n)(\pi(1 - t)(t + t^n) - (1 - \pi)(u'(\delta)/\phi) z^h(t - t^n))}{t^n - nt + n - 1} + O(z^h - z^l)^2
\]

Using bond prices \( P^h_n \) and \( P^l_n \), we deduce then the following expression for the coefficient

\[
\beta_n = (n - 1) \frac{-(\pi^h + \pi^l - 1) \frac{1}{n} \ln \frac{P^n_{h-1}}{P^n_{l-1}} + \frac{1}{n} \ln \frac{P^n_{h}}{P^n_{l}}}{-\frac{1}{n} \ln \frac{P^n_{h}}{P^n_{l}} + \ln \frac{P^n_{l}}{P^n_{h}}}
\]

F.1.1 Technical preliminaries

Expression of the price ratio. Noting \( X_k = \frac{C^h_k}{z^h} \), we have the following relationship:

\[
X_k = f(X_{k-1}) \quad \text{and} \quad f(X) = \frac{z^h}{z^l} \frac{\pi^h \alpha z^l (1 - \alpha)(u'(\delta)/\phi) z^h}{\pi^l + (1 - \alpha)(u'(\delta)/\phi) z^h} X + 1 - \pi^h \]

If we note \( f^{(k)} = f \circ f^{(k-1)} \) and \( f^{(0)} = id \), and if we remark that \( X_0 = 1 \), we obtain:

\[
X_k = f^{(k)}(1) \quad \text{and} \quad \beta_n = (n - 1) \frac{-(\pi^h + \pi^l - 1) \frac{1}{n} \ln \frac{f^{(n-1)}(1)}{f^{(n)}(1)} + \frac{1}{n} \ln f^{(n)}(1)}{-\frac{1}{n} \ln f^{(n)}(1) + \ln f^{(1)}(1)}
\]

Second order approximations With obvious notations, we express \( \beta_n \) as \( \beta_n = (n - 1) \frac{\Delta_n}{\Delta_n} \).

Because \( \Gamma_n|_{z^l=x^h} = \Delta_n|_{z^l=x^h} = 0 \), we consider second order approximations around \( z^l = z^h \):

\[
\beta_n \approx (n - 1) \frac{\partial z^l \Gamma_n|_{z^l=x^h}}{\partial z^l \Delta_n|_{z^l=x^h}} \left(1 - \frac{z^h - z^l}{2} \left( \partial^2_{z^l z^l} \Gamma_n|_{z^l=x^h} - \partial^2_{z^l z^l} \Delta_n|_{z^l=x^h} \right) \right) + O(z^h - z^l)^2
\]

Remarking that \( \forall k \quad f^{(k)} = 1 \), preceding partial derivatives express as functions of \( f^{(n)} \):

\[
\partial^2_{z^l z^l} \Delta_n = -\frac{1}{n} \frac{\partial f^{(n)}}{\partial z^l} (1) + \frac{\partial f^{(1)}}{\partial z^l} (1)
\]

\[
\partial^2_{z^l z^l} \Gamma_n = \frac{1}{n} \frac{\partial f^{(n)}}{\partial z^l} (1) - \frac{\pi^h + \pi^l - 1}{n - 1} \frac{\partial f^{(n-1)}}{\partial z^l} (1)
\]

\[
\partial^2_{z^l z^l} \Gamma_n = \frac{1}{n} \left( \frac{\partial^2 f^{(n)}}{\partial z^l} (1) - \left( \frac{\partial f^{(n)}}{\partial z^l} (1) \right)^2 \right) - \frac{\pi^h + \pi^l - 1}{n - 1} \left( \frac{\partial^2 f^{(n-1)}}{\partial z^l} (1) - \left( \frac{\partial f^{(n-1)}}{\partial z^l} (1) \right)^2 \right)
\]
**Partial derivatives** In order to express $\beta_n$, we need analytical expressions of partial derivatives of $f^{(n)}$ relative to $z^l$.

**Derivative relative to $z^l$** We derive a tractable expression of the derivative of $f^{(n)}(1)$ relative to $z^l$. Because $f^{(n)}$ is defined recursively, we need to use the derivative of $f$ relative to $X$, which we note $\frac{\partial f}{\partial X}$. We also need the expression of the derivative relative to $z^l$, which is denoted $\frac{\partial f}{\partial z^l}$. By inference, it is straightforward to prove that the derivative of $f^{(n)}(X)$ relative to $z^l$ expresses as:

$$\frac{\partial}{\partial z^l} f^{(n)}(X) = \sum_{k=0}^{n-1} \left( \frac{\partial f}{\partial z^l} \circ f^{(k)}(X) \right) \prod_{j=k+1}^{n-1} \left( \frac{\partial f}{\partial X} \circ f^{(j)}(X) \right)$$

Partial derivatives simplify in the neighborhood of $z^h = z^l = z$ into:

$$\forall k \geq 1 \quad f^{(k)}(1) = 1$$

$$\frac{\partial f}{\partial z^l}(1) = -\frac{1}{z} \frac{\alpha(2 - \pi^h - \pi^l) + (1 - \alpha)(u'(\delta)/\phi) z}{\alpha + (1 - \alpha)(u'(\delta)/\phi) z} < 0$$

$$\frac{\partial f}{\partial X}(1) = \pi^h + \pi^l - 1$$

The derivative of $f^{(n)}$ relative to $z^l$ expresses as:

$$\frac{\partial}{\partial z^l} f^{(n)}(1) = \frac{\partial}{\partial z^l} \ln f^{(n)}(1) = -\frac{1}{z} \frac{\alpha(2 - \pi^h - \pi^l) + (1 - \alpha)(u'(\delta)/\phi) z}{\alpha + (1 - \alpha)(u'(\delta)/\phi) z} \sum_{k=0}^{n-1} \prod_{j=k+1}^{n-1} (\pi^h + \pi^l - 1)$$

$$= -\frac{1}{z} \frac{\alpha(2 - \pi^h - \pi^l) + (1 - \alpha)(u'(\delta)/\phi) z}{(2 - \pi^h - \pi^l)(\alpha + (1 - \alpha)(u'(\delta)/\phi) z)} \left(1 - (\pi^h + \pi^l - 1)^n\right) \leq 0$$

For sake of simplicity, we introduce the notation $t$:

$$t = \pi^h + \pi^l - 1 \in [0, 1]$$

The first derivative of $f^{(n)}$ relative to $z^l$ expresses as:

$$\frac{\partial}{\partial z^l} f^{(n)}(1) = -\frac{1}{z} \frac{\alpha + (1 - \alpha)(u'(\delta)/\phi) z}{\alpha + (1 - \alpha)(u'(\delta)/\phi) z} (1-t^n)$$

$$= \kappa_0 > 0$$
Second order derivative \( \frac{\partial^2 f^{(n)}}{(\partial z)^2} (1) \) We derive an analytical expression for the second order derivative \( \frac{\partial^2 f^{(n)}}{(\partial z)^2} (1) \) at \( z^l = z^h \). First, the second order derivative expresses as:

\[
\frac{\partial^2 f^{(n)}}{(\partial z)^2} (X) = \sum_{k=0}^{n-1} \frac{\partial f^{(k)}}{(\partial z)^k} (X) \frac{\partial f^{(n-k)}}{(\partial z)^{n-k}} (X) \frac{\partial^2 f^{(n)}}{(\partial z)^2} (X) \]

For reading convenience, we give the expressions of partial derivatives only for \( X = 1 \) and \( z^h = z^l = z \):

\[
\begin{align*}
\partial_X f \left( f^{(j)} (1) \right) & = \pi^h + \pi^l - 1 \\
\partial_z f \left( f^{(j)} (1) \right) & = -\frac{1}{z} \left( (\alpha - \pi^h) + (1 - \alpha)(u'(\delta)/\phi) z \right) \\
\partial_z^2 f \left( f^{(j)} (1) \right) & = \frac{2}{z} \left( (\alpha - \pi^h + \pi^l) (1 - \alpha)(u'(\delta)/\phi) z + (1 - \pi^h)(\pi^l - 1) \right) \\
\partial_X^2 f \left( f^{(j)} (1) \right) & = -\frac{1}{z} \left( (\alpha - \pi^h + \pi^l) (1 - \alpha)(u'(\delta)/\phi) z \right) \\
\partial_X^3 f \left( f^{(j)} (1) \right) & = -2(1 - \pi^l)(\pi^h + \pi^l - 1)
\end{align*}
\]

After some rearrangement, we obtain the following expression for the second order derivative:

\[
\frac{\partial^2 f^{(n)}}{(\partial z)^2} (1) = \frac{2}{z^2 (\alpha + (1 - \alpha)(u'(\delta)/\phi) z)^2 (1 - t)} \times \left\{ (1 - t^n) \left( \alpha^2 (1 - \pi^h) t + (1 + t - t^n) \frac{1 - \pi^l}{(1 - t)^2} \right) (\alpha(1 - t) + (1 - \alpha)(u'(\delta)/\phi) z)^2 \right. \\
+ (\pi^h - \pi^l) (\alpha(1 - t) + (1 - \alpha)(u'(\delta)/\phi) z) (1 - \alpha)(u'(\delta)/\phi) z \left. \frac{t}{(1 - t)^2} \left( -1 + nt^n - (n - 1)t^n \right) \right\}
\]

We are now able to express \( \frac{\partial^2 f^{(n)}}{(\partial z)^2} (1) \) as useful for derivatives of \( \Delta_n \) and \( \Gamma_n \). We define \( \kappa_{\alpha,t} = \alpha + (1 - \alpha)(u'(\delta)/\phi) t \). After some manipulation, the expression
becomes:

$$\frac{\partial^2 f^{(n)}}{(\partial z')^2}(1) - \left(\frac{\partial f^{(n)}}{\partial z'}(1)\right)^2 = \frac{2}{z_h(1-t)(\alpha + (1-\alpha)(u'(\delta)/\phi) z^h)^2}$$

$$\times \left\{ (\alpha^2 (1 - \pi^h) t + (1 - t - (\pi^h + \pi^l)(1 - \pi^l)) \kappa_{\alpha,t}^2 + (\pi^h - \pi^l) \kappa_{\alpha,t} \alpha t)(1 - t^n) \right\} \kappa_{\alpha,t} = \kappa_{n,t} (1 - t^n)$$

$$\beta = \kappa_1 (1 - t^n) + \kappa_2 n t^n + \kappa_3 (1 - t^n)^2$$

**F.1.2 Expression of \( \beta_n \)**

Using preceding derivatives, we are now able to derive an analytical expression of the regression coefficient \( \beta_n \). We call back the first order approximation of \( \beta_n \):

$$\beta_n \approx (n - 1) \frac{\partial z_i \Gamma_n}{\partial z_i \Delta_n} \left( 1 - \frac{z^h - z^l}{2} \left( \frac{\partial z_i \Delta_n}{\partial z_i \Gamma_n} - \frac{\partial z_i \Delta_n}{\partial z_i \Delta_n} \right) \right) + O(z^h - z^l)^2$$

Using the preceding technical preliminaries, the ratio \( \frac{\partial z_i \Gamma_n}{\partial z_i \Delta_n} \) expresses as:

$$\frac{\partial z_i \Gamma_n}{\partial z_i \Delta_n} = \frac{1 - t^n - t \frac{t^{-n-1}}{n-1}}{-1 - t^n + 1 - t} = \frac{1}{n - 1} \Rightarrow \beta_n \approx 1 - \frac{z^h - z^l}{2} \left( \frac{\partial z_i \Delta_n}{\partial z_i \Gamma_n} + \frac{\partial z_i \Delta_n}{\partial z_i \Delta_n} \right) + O(z^h - z^l)^2$$

Both other ratios expressions are:

$$\frac{\partial^2 z_i \Delta_n}{\partial z_i \Gamma_n} = \frac{\kappa_1}{\kappa_0} + \frac{\kappa_2 (-t^n + t) + \kappa_3 \left( -\frac{(1-t^n)^2}{n} + (1-t)^2 \right)}{\kappa_0 \left( -\frac{1-t^n}{n} + 1 - t \right)}$$

$$\frac{\partial^2 z_i \Gamma_n}{\partial z_i \Delta_n} = \frac{\kappa_1}{\kappa_0} + \frac{\kappa_3 \left( \frac{(1-t)^2}{n} - t \frac{1-t^{-n-1}}{n-1} \right)}{\kappa_0 \left( 1 - t^n - t \frac{1-t^{-n-1}}{n-1} \right)}$$

With:

$$\frac{\kappa_2}{\kappa_0} = \frac{2(\pi^h - \pi^l)}{z_h(1-t)(\alpha + (1-\alpha)(u'(\delta)/\phi) z^h)(1-\alpha)(u'(\delta)/\phi) z}$$

$$\frac{\kappa_3}{\kappa_0} = \frac{2(\pi^h - \pi^l)}{z_h(1-t)(\alpha + (1-\alpha)(u'(\delta)/\phi) z^h)}$$

The difference between both ratios expresses as:

$$\frac{\partial^2 z_i \Gamma_n}{\partial z_i \Delta_n} - \frac{\partial^2 z_i \Delta_n}{\partial z_i \Gamma_n} = \frac{\kappa_3 n(1-t)(t - t^{2n-1})}{\kappa_0 n^2 - n + 1 - t} - \frac{\kappa_2 n t (1 - t^{n-1})}{\kappa_0 t^n - n t + n - 1}$$
We finally deduce a first order approximation for $\beta_n$:

$$
\beta_n \approx 1 + \frac{\left( z^h - z^l \right) (\pi^h - \pi^l)}{2 \ z^h (1-t) t (\alpha + (1-\alpha) (u'(\delta)/\phi) z^h) n (t-t^n) \ (\alpha(1-t)(t+t^n) - (1-\alpha)(u'(\delta)/\phi) z^h (t-t^n))}{t^n - n t + n - 1} + O(z^h - z^l)^2
$$

F.2 Proof of Proposition 4

We use the preceding first order approximation of $\beta_n$ to derive our results.

**Role of credit constraints on $\beta_n$**

To assess the impact of credit constraints on $\beta_n$, we compute the derivative $\partial_\alpha \beta_n$:

$$
\partial_\alpha \beta_n = \frac{z^h - z^l}{2 \ z^h} \ \frac{n (\pi^h - \pi^l) (t-t^n) (2-t-t^n) u z^h}{(1-t) (t^n - n t + n - 1) (\alpha + (1-\alpha) (u'(\delta)/\phi) z^h)^2} \geq 0
$$

As long as $\pi^h > \pi^l$, $\beta_n$ increases with $\alpha$ and therefore decreases with credit constraints.

**Size and Monotony of $\beta_n$.**

First, from the expression of $\beta_n$, it is straightforward to deduce that:

$$
\beta_n \leq 1 \ \text{if} \ \alpha(1-t)(t+t^n) \leq (1-\alpha)(u'(\delta)/\phi) z^h (t-t^n)
$$

Whereas without credit constraints ($\alpha = 1$) the regression coefficient $\beta_n$ is always strictly larger than 1, the introduction of credit constraints is likely to make it smaller than 1 and possibly negative.

Second, to prove the monotony of $\beta_n$, we compute the difference $\beta_{n+1} - \beta_n$:

$$
\beta_{n+1} - \beta_n = - \frac{\left( z^h - z^l \right) (\pi^h - \pi^l)}{2 \ z^h (1-t) (\alpha + (1-\alpha) (u'(\delta)/\phi) z^h) (t^n - n t + n - 1) (t^{n+1} - (n+1) t + n)} \times
$$

$$
\left\{ \begin{array}{l}
\alpha (1-t) \ (1+n t^{n-1} - (n+1) t^n) \ (1-n t^{n-1} + n t^{n+1} - t^{2n}) \ = S_1^\alpha(t) \\
-(1-\alpha) (u'(\delta)/\phi) z^h \ (1-2 n^2 t^{n-1} + (4 n^2 + n - 3) t^n - (n^2 - 3) t^{2n} - (n+1) t^{3n}) \\
-(n+2 n^2) t^{1+n} + n^2 t^{-2+2n} - (n+2 n^2) t^{-1+2n} + (n+2 n^2) t^{1+2n} + n t^{-1+3n} \ = S_2^\alpha(t) 
\end{array} \right.
$$
The coefficient $\beta_n$ is decreasing or $\beta_{n+1} \leq \beta_n$ if:

$$\alpha (1 - t) S^1_n(t) \geq (1 - \alpha) \frac{u'(\delta)}{\phi} z^h S^2_n(t)$$

To have $\beta_n$ smaller than 1 and a decreasing pattern, coefficients must verify:

$$S^2_n(t) \leq \frac{\alpha (1 - t)}{(1 - \alpha) \frac{u'(\delta)}{\phi} z^h} S^1_n(t) \leq S^1_n(t) \frac{t - t^n}{t + t^n}$$

It is noteworthy that both conditions are compatible in the sense that for all $n$, $S^1_n(t) \frac{t - t^n}{t + t^n} > S^2_n(t)$. It is therefore possible to simultaneously have $\beta_n < 1$ and a decreasing pattern.

### G Proof of proposition 5

We prove that: (i) ex ante welfare increases with bond supply and (ii) at date 0, the welfare of $eu$ and $uu$ goes only if the discount factor $\beta$ is large enough. We suppose that there is no aggregate shock and $z_t = 1$. We note $U$ the vector of instantaneous utility. Using budget constraints, it expresses as:

$$U = \begin{bmatrix} u(c^{ee}) - l^{ee} \\ u(c^{ue}) - l^{ue} \\ u(c^{eu}) - l^{eu} \\ u(c^{uu}) - l^{uu} \end{bmatrix} = \begin{bmatrix} u^t(1) - 1 \\ u^t(1) - 1 - \sum_{k=1}^n C_k B_k \\ u(\delta + \sum_{k=1}^n C_k B_k) \\ u(\delta) \end{bmatrix}$$

The transition matrix $\Omega$ for the four states $\{ee, ue, eu, uu\}$ of the economy is the following:

$$\Omega = \begin{bmatrix} \alpha & 0 & 1 - \alpha & 0 \\ \alpha & 0 & 1 - \alpha & 0 \\ 0 & 1 - \rho & 0 & \rho \\ 0 & 1 - \rho & 0 & \rho \end{bmatrix} = Q.D.Q^{-1} \text{ with } Q = \begin{bmatrix} 1 & 1 - \alpha & 0 & 1 - \alpha \\ 1 & 0 & \rho & 1 - \alpha \\ 1 & -\alpha & 0 & -(1 - \rho) \\ 1 & 0 & -(1 - \rho) & -(1 - \rho) \end{bmatrix}$$

and $D = \text{Diag}(1, 0, 0, \alpha + \rho - 1)$

The vector $U$ of the four intertemporal utilities is $U = \sum_{k=0}^{\infty} \beta^k \Omega^k U = \sum_{k=0}^{\infty} \beta^k Q D^k Q^{-1} U$
The impact of a bond increase $B_k$ on this intertemporal utility $U$ is:

$$\frac{\partial U}{\partial B_k} = \begin{bmatrix} \frac{\partial u^e}{\partial B_k} \\ \frac{\partial u^u}{\partial B_k} \\ \frac{\partial u^u}{\partial B_k} \\ \frac{\partial u^u}{\partial B_k} \end{bmatrix} = C_{k-1} Q \begin{bmatrix} \frac{1}{1-\beta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\beta(\alpha+\rho-1)} \end{bmatrix} Q^{-1} \begin{bmatrix} 0 \\ 0 \\ u' (\delta + \sum_{k=1}^n C_{k-1} B_k) \\ 0 \end{bmatrix}$$

First, the first part of the proposition stems directly from the impact of bond supply on instantaneous utility. Whereas in the states $ee$ and $uu$, the utility does not change, it decreases from $-1$ in $ue$ and goes up from $u' = u'(\delta + \sum_{k=1}^n C_{k-1} B_k)$ in $eu$. Since our equilibrium exists only if $u' = u'(\delta + \sum_{k=1}^n C_{k-1} B_k) > 1$, and that $eu$ and $ue$ states are equally probable, ex ante welfare always increases with bond supply.

To obtain the second part of the result, we expand the preceding expression of $\frac{\partial U}{\partial B_k}$:

$$\frac{\partial U}{\partial B_k} = \frac{C_{k-1}}{(2-\alpha-\rho)(1-\beta)(1-\beta(\alpha+\rho-1))} \begin{bmatrix} \beta (1-\alpha)(u' - \beta(1+\rho(u' - 1))) \\ (1-\beta\rho)(\beta(\alpha + (1-\alpha)u') - 1) \\ (1-\beta\alpha)(u' - \beta(1+\rho(u' - 1))) \\ \beta(1-\rho)(\beta(\alpha + (1-\alpha)u') - 1) \end{bmatrix}$$

Noting $\beta_0 = [\alpha + (1-\alpha)u' (\delta + \sum_{k=1}^n C_{k-1} B_k)]^{-1}$, it is straightforward to prove that if $\beta > \beta_0$, the expected welfare of all agents’ types increases with $B_k$ and if $\beta > \beta_0$, that the welfare of $ue$ and $uu$ decreases.