About the choice between a reversed multi-attribute auction and a reversed auction with a quality threshold

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Abstract

We consider a reversed English Auction model with a 2-dimensional utility function: the buyer cares about the price and the quality of a single and indivisible good or service. The buyer may choose to conduct a Multi-Attribute Auction - MAA - and to reveal her utility function and the parameters of the functional form. Alternatively, she may choose to conduct a Reversed Auction with a Quality threshold - RAQ. In the latter, she reveals a minimum level of quality $K$ to select a subset of qualified bidders and then organizes a reversed auction based on price. These two implementations significantly differ in the information revealed. The question raised is to quantify the expected utility loss of choosing RAQ instead of MAA. Moreover, we determine values of $K$ that minimize this cost to the buyer. Finally, we compare the efficiency of the implementation of a RAQ with that of a MAA. We restrict our analytical approach to the case where only two bidders participate in the auction. Then, we use simulations to extend our results when more than two bidders are qualified.
1 Introduction

The development of auction theory in recent years has brought forward the discussion on multi-attribute auctions. These attributes, that are for instance quality, delivery time, lead time or terms of payment, intervene in the final decision and can be non-price attributes. In a deal that is composed of such multiple dimensions, there are two primary objectives in the auction theory literature - allocation efficiency and utility maximization. Efficiency is important when it is necessary to both allocate the object to those who value it the most and maximize the total surplus. This objective is often in conflict with that of utility maximization since the buyer may increase her utility solely by allocating the object inefficiently. Independently from this conflict, a key issue in MAA is the information revelation. The buyer has to announce a scoring rule in terms of the bid price and the quality attribute. Therefore, the aspect we focus on in this paper is whether a buyer who cares about both a quality criterion and a price criterion has to reveal all her private information to bidders or has to first perform a selection among the bidders based on a minimum quality level criterion and then conduct on a reversed English auction on price. These two mechanisms significantly differ in the information revealed. The buyer may have an incentive to hide strategic information that could be later used by competitors later. The question raised is to quantify the expected utility loss of choosing a RAQ instead of a MAA based on the utility maximization objective. The two following examples highlight that the choice between the two different implementations is crucial from the buyers’ perspective.

Two introductory examples. We start describing the environment through two introductory examples to highlight how relevant the question is in many different situations.

Example 1: a high-end shop delivers fresh products (e.g. fruits and vegetables) that has built his reputation on quality. Information on the quality requirement the shop imposes to her providers is thus a strategic key to his business. If final consumers or competitors learn that, in reality, the quality of products is not a primary factor for the shop, it might lower its market shares. Therefore, the margin she has on products can be considered as private information that has to remain private regarding competition concerns. Then, the shop faces a trade-off between revealing her true preferences as well as core business information or concealing her true cost structures even if it involves a revenue loss.

Example 2: a firm B wants to ship a container from one point to another and faces some delivery time issues. Under capacity and/or technological constraints, a subset of transporters are able to perform the task. Then, B faces a choice: on the one hand, B can reveal the parameters of her utility function, namely the impact of the delivery time in her valuation and chooses among the bids. On the other hand, she can reveal a minimum level of desired delivery time K and, in a second step, choose among the remaining bids.

Related Literature. Thiel [19], Che [9] and Branco [8] were among the first to consider auction mechanisms that both integrate price and quality attributes. They referred them to as multidimensional auctions. Later Bichler et al. [4] motivated and illustrated the use of multiattribute auctions in procurement [5] and financial [6] contexts. Parallel to these works, Teich et al. [17] introduced and criticized different multiple issue auctions and market algorithms. This field which is at the crossroads of economics, electronic commerce and operational research has received a lot of attention during the last ten years. Among others, let us mention the works of Beil et al. [2], Bichler et al. [7], David et al. [11], Koppius [13], Parkes et al. [15], Strecker et al. [16], Bellosta et al. [3] and Chen-Ritzo et al. [10]. The interested reader is referred to the recent article of Teich et al. [18] for a detailed review of the field.

From an economic perspective, Klemperer [12] proposed an Anglo-Dutch auction in which “the buyer begins by running an ascending auction until just two bidders are willing to pay the current asking price.
That is, the price is raised continuously until all but two bidders have dropped out. The two remaining bidders are then each required to make their best and final sealed-bid offer that is not lower than the current asking price, and the winner pays his bid.” This procedure allows for a revenue maximizer auction that induces a more efficient auction through the learning process of the ascending auction. In comparison, the RAQ mechanism that is considered in the paper works in an opposite way. Indeed, a subset of bidders is first qualified on the basis of a fixed quality threshold and then a traditional reversed English auction is organized. From a pure efficient auction viewpoint, Milgrom [14] has shown that efficiency is achieved if the buyer announces the true utility function of the buyer as the scoring rule, and conducts a Vickrey auction based on the resulting scores. In this paper, we will analyze the expected payoff difference from a utility maximization perspective and then bring the discussion on efficiency aspects. The interested reader is referred to the recent article of Asker et al. [1] for a description of the optimal 2-bidder auction as a practical benchmark for scoring auctions when price and quality matter.

The paper is organized as follows. Section 2 describes the model and the auction rules. The payoff and the optimal \( K \) for the buyer are analyzed in Section 3. Section 4 embeds the efficiency dimension. Section 5 concludes and carries out the discussion on the criteria for the quality threshold \( K \).

2 The model

Payoff structure. Starting from Example 2, we consider a buyer \( B \) who wants to acquire an indivisible good or service for which there are \( N \geq 2 \) risk-neutral bidders - namely the transporters.

The attributes of an offer from bidder \( i \) is a pair composed of a price attribute \( p \) and a quality attribute \( q \). For the sake of simplicity, the attributes are both defined on \([0,1]\). Consider the payoff function of buyer \( B \):

\[
\pi_B = u_B(q) - p
\]

The payoff function of the buyer is defined as the utility \( u(q) \) she gets from the quality in monetary units minus the cost \( p \) of the winning bid. In addition, we assume that \( u(q) \) is linear and strictly increasing in quality. Therefore, \( u(q) = \alpha q \) where \( \alpha \in [0, +\infty[ \). \( \alpha \) represents the relative weight given to the quality with respect to the final price in the payoff function of the buyer. For instance, a value of 1 means that \( B \) awards the same importance to the price and the quality of the good.

The payoff function of bidder \( i \) is

\[
\pi_i = p - v_i(q)
\]

The payoff function of the winning seller \( i \) is the final price \( p \) minus his costs \( v_i(q) \). Note that no assumption so far is made on the cost function as it will have no impact in the auction. Furthermore, we will assume that any bidder that is able to provide a quality level \( q \) is always able to provide a worse quality.

Information. The number of bidders and the functional form of the buyer are common knowledge among transporters and the buyer. Signals are privately observed by each transporter. They are identically and independently distributed over \([0, 1]\). Furthermore, this distribution is common knowledge. In addition, we consider as a working assumption that there is no correlation between price and quality.

Assumption 1. \((q, p)\) follows a joint density distribution \( f_{q,p} \). Quality and price are independent variables.
Auction rules. We consider two distinct implementations of a "reversed English-Button-Auction". The first implementation is a Multi-Attribute Auction - MAA - implementation. In this implementation, we assume that the utility function of the buyer, i.e. the functional form, is common knowledge as well as the parameter $\alpha$.

The second implementation is a Reversed Auction with a Quality threshold - RAQ - implementation organized as a 2-step auction. The sequence is the following: in the first step, a minimum quality requirement $K$ is revealed by the buyer to the bidders. To fulfill the minimum requirement is a qualifier for the second step in which the remaining bidders enter in a descending price auction.

For example, consider two bidders $i = 1, 2$. From the dynamics of the auction, we have to take into account three different cases:

- All bidders are qualified for the second stage of the implementation;
- No bidder fulfills the minimum quality requirement $K$;
- There is only one bidder that matches the minimum requirement $K$.

In both implementations, the winning bid is the lowest price-offer but the buyer pays the second minimum bid. All bidders are initially active and they remain so until they drop out irretrievably. Ties in the second round are broken by assigning the object randomly and with equal probability to one of the lowest bidders. In the next section, we thus bring the analysis on the trade-off between revealing $\alpha$ or $K$.

3 Payoff Analysis

First we characterize the expected payoff in the MAA implementation and then in the RAQ implementation in order to compare the cost to the buyer in hiding information induced by the two implementations in a state space $(\alpha, K)$.

3.1 MAA implementation

From Assumption 1, we first derive the probability density function (pdf) with respect to $\pi_B$, $f_{\pi_B}$. Denote $F_{\pi_B}(t) = \int_{-\infty}^{t} f_{\pi_B}(v)dv$ as the corresponding distribution function to the payoff of firm $B$ such that

$$F_{\pi_B}(t) = P(\pi_B \leq t) = P(\alpha q - p \leq t) = \int_{aq - p \leq t} f_{qp}(q,p)dqdp$$

Then, we proceed to a change of variables. Let $v = \alpha q - p$ and $w = p$. Thus, $q = \frac{v + w}{\alpha}$, $p = w$ and the determinant of the jacobian is $|J| = \begin{vmatrix} \frac{1}{\alpha} & 1 \\ 0 & \frac{1}{\alpha} \end{vmatrix} = \frac{1}{\alpha}$. Therefore, the cumulative distribution function of the payoff function can be rewritten with respect to $v$ and $w$

$$F_{\pi_B}(t) = \int_{-\infty}^{t} f_{\pi_B}(v)dv = \int_{-\infty}^{t} dv \int_{-\infty}^{\infty} f_{qp}(\frac{v + w}{\alpha}, w) \frac{1}{\alpha} dw$$

3
We can finally characterize the pdf \( f_{nq}(v) \) with respect to \( w \). Based on Assumption 1, we get

\[
f_{nq}(v) = \int_{-\infty}^{\infty} f_q\left(\frac{v + w}{\alpha}, w\right) \frac{1}{\alpha} dw = \int_{-\infty}^{\infty} f_q\left(\frac{v + w}{\alpha}\right) f_p(w) \frac{1}{\alpha} dw
\]

(3)

As the density function of \( q \) is different from 0, \( f_q\left(\frac{v + w}{\alpha}\right) \neq 0 \). It follows that \( \frac{v + w}{\alpha} \in [0, 1] \iff w \in [-v, (\alpha - v)] \). From this, we disentangle two cases: when \( \alpha \geq 1 \) (i.e. the case where the quality is relatively more important than the price in the utility function) and the opposite, \( \alpha \leq 1 \). If \( \alpha \geq 1 \), the interior \([0, 1]\) is greater or equal to \([0, 1]\) and smaller or equal otherwise. We report the former case here and the latter in the Appendix.

\( \alpha \geq 1 \) case. The five following cases arise:

- \( v < -1, f_{nq} = 0 \);
- \( -1 \leq v < 0 \), thus the pdf is \( \int_{-v}^{0} \frac{1}{\alpha} dw = \frac{1-v}{\alpha} \);
- \( 0 \leq v < \alpha - 1 \), thus the pdf is \( \int_{0}^{1} \frac{1}{\alpha} dw = \frac{1}{\alpha} \);
- \( \alpha - 1 \leq v < \alpha \), thus the pdf is \( \int_{0}^{\alpha-v} \frac{1}{\alpha} dw = \frac{\alpha-v}{\alpha} \);
- \( v \geq \alpha, f_{nq} = 0 \)

Second, we need to characterize \( F_{nq} \) in each of the different cases given above.

- \( v < -1, F_{nq}(v) = 0 \);
- \( -1 \leq v < 0, F_{nq}(v) = 0 + \int_{-v}^{0} \frac{1}{\alpha} dx = \frac{1}{\alpha} (v + 1)^2 \);
- \( 0 \leq v < \alpha - 1, F_{nq}(v) = \frac{1}{\alpha} (v + 1)^2 |_{v=0} + \int_{0}^{v} \frac{1}{\alpha} dx = \frac{1}{\alpha} + \frac{v^2}{2\alpha} \);
- \( \alpha - 1 \leq v < \alpha, F_{nq}(v) = \frac{1}{\alpha} + \frac{1}{\alpha} |_{v=\alpha-1} + \int_{\alpha-1}^{v} \frac{\alpha-x}{\alpha} dx = 1 + v - \frac{\alpha^2}{2\alpha} = \frac{v^2}{2\alpha} \);\( v \geq \alpha, F_{nq}(v)(v = \alpha) = 1 \);

Third, we are now ready to derive the joint cumulative distribution function (cdf) and finally the pdf. Let \( X \) and \( Y \) be two variables of distribution \( F_X \) and \( F_Y \) respectively. Denote \( Z = \min\{X, Y\} \). \( Z \) is thus the minimum utility the firm \( B \) receives as the outcome of the second implementation. The cdf of \( Z \) is thus \( F_Z(t) = P(Z \leq t) = P(\min\{X, Y\} \leq t) = 1 - P(\min\{X, Y\} > t) = 1 - P(X > t \text{ and } Y > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - F_X(t))(1 - F_Y(t)) \).

**Assumption 2.** \( X \) and \( Y \) are IID. Therefore, \( F_X(t) = F_Y(t) = F(t) \)
\[ F_Z(t) = 1 - (1 - 2F(t) + F^2(t)) = F(t)(2 - F(t)) \] (4)

Therefore, the pdf \( f_Z(t) \) is

\[ f_Z(t) = f(t)(2 - F(t)) + F(t)(-f(t)) = 2f(t)(1 - F(t)) \] (5)

We summarize the different cases for \( F_{\pi B} \), \( f_{\pi B} \) and \( f_Z \) in the following table:

<table>
<thead>
<tr>
<th>( v )</th>
<th>( f_{\pi B}(v) )</th>
<th>( F_{\pi B}(v) )</th>
<th>( f_Z(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v &lt; -1 )</td>
<td>0</td>
<td>( \frac{1}{\alpha} )</td>
<td>( \frac{1}{\alpha} )</td>
</tr>
<tr>
<td>( -1 \leq v &lt; 0 )</td>
<td>( \frac{1}{\alpha v} (v + 1)^2 )</td>
<td>( \frac{1}{\alpha v} + \frac{v}{\alpha} )</td>
<td>( 1 + v - \frac{v^2}{\alpha} - \frac{v}{\alpha} )</td>
</tr>
<tr>
<td>( 0 \leq v &lt; \alpha - 1 )</td>
<td>( \frac{\alpha - v}{\alpha} )</td>
<td>( \frac{\alpha - v}{\alpha} [2\alpha - (v + 1)^2] )</td>
<td>( \frac{\alpha - v}{\alpha} [2\alpha - 1 - 2v] )</td>
</tr>
<tr>
<td>( \alpha - 1 \leq v &lt; \alpha )</td>
<td>( \frac{\alpha - v}{\alpha} )</td>
<td>( \frac{\alpha - v}{\alpha} (2\alpha - 1 - 2v) )</td>
<td>( \frac{\alpha - v}{\alpha} (\alpha - v)^2 )</td>
</tr>
<tr>
<td>( v \geq \alpha )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Finally, we can compute the expected payoff of the buyer \( B \). The three relevant cases are

(a) \[ \frac{1}{\alpha^2} \int_{-1}^{0} (v + 1)[2\alpha - (v + 1)^2]vdv = \frac{1}{60\alpha^2}(3 - 20\alpha) \]

(b) \[ \frac{1}{\alpha^2} \int_{0}^{\alpha - 1} (2\alpha - 1 - 2v)[2\alpha - (v + 1)^2]vdv = \frac{(\alpha - 1)^2}{6\alpha^2}(2\alpha + 1) \]

(c) \[ \frac{1}{\alpha^2} \int_{\alpha - 1}^{\alpha} (\alpha - v)^3vdv = \frac{1}{20\alpha^2}(5\alpha - 4) \]

Thus, (a)+(b)+(c) is

\[ E[\pi_B] = \frac{1}{60\alpha^2}(20\alpha^3 - 30\alpha^2 - 5\alpha + 1) \] (6)

### 3.2 RAQ implementation

We first present the buyer’s expected payoff when bidders offer a quality equal to the minimum requirement \( K \) and then allow bidders to offer a higher quality. The motivation for the latter is of course that a producer or a company offering services may not perfectly adjust the production to the threshold \( K \).

#### 3.2.1 Case \( q = K \)

The auction mechanism consists in a 2-step auction: the first step is on quality and the second on price. The Figure 1 represents all of the cases when 2 bidders are qualified for the last step. In this framework, three different cases (and the associated probabilities) may occur:

\[ ^1 \text{when } \alpha < 1, E[\pi_B] = \frac{1}{60}(\alpha^3 - 5\alpha^2 + 30\alpha - 40) \]
• **Case 1:** more than 1 bidder is qualified for the auction. In Figure 1 panel (a), the two offers in terms of quality are better than the minimum requirement $K$. The probability of such an event is:

$$P(q_1 > K \text{ and } q_2 > K) \equiv P(q_1 > K)P(q_2 > K) = [1 - F_q(K)]^2 = (1 - K)^2$$

The payoff is $\pi_B = \alpha K - p_{(2)}$ where $p_{(2)} = \text{Max}(p_1, p_2)$.

Taking the expected payoff thus requires the first moment from $p_{(2)}$: $E(p_{(2)}) = \int_0^1 t.dP[\text{Max}(p_1, p_2) \leq K] = \int_0^1 t.2tdt = \frac{3}{2}$.

• **Case 2:** only one bidder is qualified for the auction in panel (b). The probability of such an event is:

$$P(q_1 \leq K \text{ and } q_2 \leq K) \equiv 1 - (1 - K)^2 - K^2 = 2K(1 - K)$$

The payoff is $\pi_B = \alpha K - 1$

• **Case 3:** no one is qualified. See panel (c) Figure 1.

The probability of such an event is:

$$P(q_1 \leq K \text{ and } q_2 \leq K) = [F_q(K)]^2 = K^2$$

The payoff is $-1$ as we consider that firm B will face a loss not having the service fulfilled (in the example, the transport will not take place and thus, firm B will face extra storage costs for example)

Summing the three possible cases, we obtain the expected payoff to the buyer $B$

$$E(\pi_B | q = K) = (\alpha K - E(p_{(2)}))(1 - K)^2 - K^2 + (\alpha K - 1)2K(1 - K)$$

Simplifying in the expression, we obtain

$$E[\pi_B | q = K] = -\alpha K^3 + \frac{K^2}{3} + K(\alpha - \frac{2}{3}) - \frac{2}{3} \quad (7)$$

### 3.2.2 Case $q > K$

In this framework, three different cases (and the associated probabilities) may occur:

• **Case 1:** more than 1 bidder is qualified for the auction. In Figure 1 panel (a), the two offers in terms of quality are better than the minimum requirement $K$. The fact that a bidder can offer a better quality imposes now that the buyer takes into account that each bidder has a fifty percent chance being the winning bidder. The expected payoff thus requires the first moment from $q_{(1)} = \text{min}(q_1, q_2), q_{(2)} = \text{Max}(q_1, q_2), p_{(2)}$.

The probability of such an event is:

$$P(q_1 > K \text{ and } q_2 > K) \equiv P(q_1 > K)P(q_2 > K) = [1 - F_q(K)]^2 = (1 - K)^2$$

The expected payoff is $E[\pi_B | q_1 > K \text{ and } q_2 > K] = \frac{1}{2}(aE[q_{(2)}] - E[p_{(2)}]) + \frac{1}{2}(\alpha E[q_{(1)}] - E[p_{(2)}])$. $E[q_{(1)}] = \int_0^1 t.dP[q_{(1)} = t|q_{(1)} > K \text{ and } q_{(2)} > K]$ where $P[q_{(1)} = t|q_{(1)} > K \text{ and } q_{(2)} > K] = \frac{P[q_{(1)} = t|q_{(1)} > K \text{ and } q_{(2)} > K]}{P[q_{(1)} > K \text{ and } q_{(2)} > K]}$. $P[q_{(1)} = t|q_{(1)} > K \text{ and } q_{(2)} > K] = \frac{P[q_{(1)} = t|q_{(1)} > K \text{ and } q_{(2)} > K]}{P[q_{(1)} > K \text{ and } q_{(2)} > K]} = \frac{t-K}{(1-K)^2}$ with $P[q_{(2)} > K] = 1$. Thus $E[q_{(1)}] = \frac{1}{(1-K)^2} \int_0^1 tdt = \frac{1}{2(1-K)^2}$.
Similarly, \( E[q_2] = \frac{1}{4} (1 - K) \). 

\[
E[q_2] = \frac{1}{4} (1 - K)
\]

\[
E[q_2] = \frac{1}{4} (1 - K)
\]

- **Case 2**: only one bidder is qualified for the auction in panel (b). The probability of such an event is: 

\[
P(q_1 > K \text{ and } q_2 \leq K) = 1 - (1 - K)^2 - K^2 = 2K(1 - K)
\]

The expected payoff is 

\[
E[\pi_2] = aE[q_2] - 1 
\]

where 

\[
E[q_2] = \frac{1}{4} (1 - K).
\]

- **Case 3**: no one is qualified. See panel (c) Figure 1.

The probability of such an event is: 

\[
P(q_1 \leq K \text{ and } q_2 \leq K) = P(q_1 \leq K)P(q_2 \leq K) = [F_q(K)]^2 = K^2
\]

The payoff is here \(-1\).

Summing the three possible cases, we obtain the expected payoff to the buyer B

\[
E(\pi_{2} | q > K) = a \frac{3 + 3K - 8K^2 + 4K^3}{8(-1 + K)^2} + K(-2 + K) - \frac{2}{3}
\]

(8)
In what follows, we restrict our analysis to the case where bidders provide a quality equal to $K$.

### 3.3 Revenue Comparison $E[\text{diff}] = E[\pi_B] - E[\pi_B|q=K]$}

Simplifying the difference between equations (6) and (7), we find for $\alpha \geq 1$

$$E[\text{diff}] = E[\pi_B] - E[\pi_B|q=K] = \alpha K^3 - \frac{1}{3}K^2 + \left(\frac{2}{3} - \alpha\right)K + \left(\frac{1}{60\alpha^2} - \frac{1}{12\alpha} + \frac{\alpha}{3} + \frac{1}{6}\right)$$

(9)

Note immediately that the difference found for the case $\alpha \leq 1$ has the same pattern and will not modify our result for optimal $K$. The result seems to be obvious at first: the expected payoff of the MAA (i.e. full information implementation) is always larger than the expected payoff extracted from the 2-round implementation. Intuitively, the result is driven by the fact that we allow for a complete adjustment to the utility function of $B$. However, a counterpart is that the buyer reveals the vector of parameters of the utility function, i.e. $\alpha$. Thus the difference in expected payoff can be seen as a premium paid to conceal sensible information on the functional form. In practice, one may choose $K$ on a "naive" criterion - the minimal expected margin between the two implementations. Therefore, we find the roots that minimize the value of $K$

$$\text{argmin}_K(E[\pi_B] - E[\pi_B|q=K]) = \frac{1 \pm \sqrt{1 - 18\alpha + 27\alpha^2}}{9\alpha}$$

First, for $\alpha < \frac{1}{6}(3 - \sqrt{6})$, the resulting $K$ for both roots is higher than 1. Second, for $\frac{1}{6}(3 - \sqrt{6}) < \alpha < \frac{1}{6}(3 + \sqrt{6})$, the roots do not exist. Note that the $E[\text{diff}]$ is monotonic on the interval $[0, \frac{1}{6}(3 + \sqrt{6})]$. Therefore, the optimal value for $K$ is equal to 0. Intuitively, $\alpha$’s realizations being small on this interval, quality is not important. Third, for $\alpha \geq \frac{1}{6}(3 + \sqrt{6})$, both roots lead to valid values for optimal $K$. The second root constitutes a minimum for $\alpha \geq \frac{(2 + \sqrt{3})}{6}$. This result shows that the selection of bidders based on a minimum quality threshold is worthwhile only if $\alpha$ exceeds a certain value. In other words, the distinction between a RAQ and a traditional reversed auction based on price makes no sense for low values of $\alpha$ but is worthwhile even for values of $\alpha$ lower than 1.

$$K = \lim_{\alpha \to \frac{(2 + \sqrt{3})}{6}} \frac{1 + \sqrt{1 - 18\alpha + 27\alpha^2}}{9\alpha} = 2 - \sqrt{3}$$

$$K = \lim_{\alpha \to \infty} \frac{1 + \sqrt{1 - 18\alpha + 27\alpha^2}}{9\alpha} = \frac{1}{\sqrt{3}}$$
3.4 Expected revenue difference for more than two bidders

Up to now, we have restricted our analysis to the case where only two bidders participate in the auction. This has allowed us to derive analytical results about the expected revenue difference between the two implementations. In this section, we consider the more general case where more than two bidders are involved in the procurement.

The expected revenue difference has been quantified on the basis of MATLAB simulations for a number of bidders ranging from 2 to 20, for $\alpha \in \{1, 1.5, 2, 2.5, 3\}$ and for values of $K$ ranging from 0 to 1 with a step of 0.1. We have run 100,000 simulations for every possible configuration. Figure 3 illustrates how the expected revenue difference evolves for different values of $K$ when the number of bidders increases. In this setting, we have considered $\alpha = 1$. Results related to other values of the parameter $\alpha$ are available in the appendix (nevertheless, conclusions related to these values remain the same as for $\alpha = 1$). The curve related to the value $K = 1$ represents an extreme situation since bidders are never qualified in the second implementation. Consequently, the associated utility is always equal to the lowest possible value, i.e. $-1$. In other words, the curve indicates how the expected revenue in a multi-attribute auction (increased by 1) is affected when the number of participants evolves. Obviously, this constitutes an upper bound for the expected revenue difference.

On the one hand, curves related to values of $K \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ show that the expected revenue difference increases with the number of bidders. This effect is straightforward since competition is based on both price and quality attributes in the first implementation while it is restricted to price (for a given minimum quality threshold) in the second implementation. When many bidders participate in the auction, it is more likely to have bids that are both attractive for the two attributes which will lead to high utility values. In the RAQ, these bids will remain stuck to lower values of the quality attribute which will lead at lower values of the utility.

On the other hand, curves related to values of $K \in \{0.6, 0.7, 0.8, 0.9\}$ show an interesting effect; when the number of bidders exceeds 2, the expected revenue difference decreases. This is due to the fact that when several bidders participate in the auction, it is more likely to have at least one of them qualified for high values of $K$. Even if competition related to the price is limited, this minimal quality value will have a direct impact on the utility values. Finally, let us note that when the number of bidders becomes more important the curve is increasing again (for instance the expected revenue difference for $K = 0.9$ and 100 bidders is equal to 0.7147). Note that the general form of the RAQ implementation for $n$ bidders is provided in Appendix C.
4 RAQ and Efficiency

In this section, we introduce an extra criterion from the buyer’s point of view. An allocation mechanism is defined as efficient when the winning bidder makes an offer that maximizes the buyer’s payoff.

Suppose that our two random variables $p, q$ are uniformly distributed on $[0, 1]$. As we defined earlier, three cases may occur: no one is qualified and there is never efficiency, one bidder is qualified or two bidders are qualified.

4.1 Probability density function from the difference of two uniform random variables

First, we need to determine the pdf from the difference of two uniform random variables on the interval $[0, 1]$. Let $V$ and $W$ be 2 random variables distributed on the interval $[0, 1]$. Let $Z = V - W$. The cdf of $Z$
is

\[ F_Z(x) = \int \int \int f_{V,W}(\xi, \eta) d\xi d\eta \]

Then, we proceed to a change of variables. Let \( \xi = u \) and \( \eta = u - v \). The determinant of the jacobian is \( |J| = 1 \). The cdf becomes

\[ F_Z(x) = \int \int f_{V,W}(u, u - v) dudv \]
\[ = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} f_{V,W}(u, u - v) du \]
\[ = \int_{-\infty}^{\infty} f_Z(u) du \]

Based on Assumption 1, the pdf of \( Z \) yields

\[ f_Z(z) = \int_{-\infty}^{\infty} f_{V,W}(u, u - v) du \]
\[ = \int_{-\infty}^{\infty} f_V(u) f_W(u - z) du \]

The two random variables are such that \( V \sim U[0,1] \) and \( W \sim U[0,1] \), we obtain

\[ f_W(u - z) = \begin{cases} 
1 & u \in [z, z + 1] \\
0 & \text{otherwise} 
\end{cases} \]

Therefore,

\[ f_Z(z) = \int_{0}^{1} f_W(u - z) du \]
\[ = \int_{[0,1]^2[z,z+1]} du \]

Finally, we find that the difference follows a pdf of a centered triangle variable such that

\[ f_Z(z) = \begin{cases} 
0 & z < -1 \\
z + 1 & -1 \le z < 0 \\
1 - z & 0 \le z \le 1 \\
0 & z > 1 
\end{cases} \]

From equation (3), we know that \( f_{\pi B}(v) = \int_{-\infty}^{\infty} f_q(v + w) f_p(w) \frac{1}{w} dw \). To recover the pdf of the buyer \( B \) for the efficiency issue, we note immediately that \( q \) and \( p \) are now two random variables linked with the difference of two dependent variables. Thus, denote \( q = \Delta q \) and \( p = \Delta p \). Let's know characterize \( f_p(v) \) and \( f_q(\frac{v+w}{\alpha}) \) when one and two bidders remain respectively.
4.2 Efficiency when 1 bidder remains ($P_{eff,1}$)

In this situation, the differences in quality and price are non negative; namely, $\Delta q$ and $\Delta p$ are both $\geq 0$. Let $p = |x - y|$ where $x$ and $y$ are two uniformly distributed random variables on $[0, 1]$. The cdf is then

$$F_{|x-y|}(t) = P(|x - y| \leq t) = P(-t \leq x - y \leq t) = P(x - y \leq t) - P(x - y \leq -t) = F_{x-y}(t) - F_{x-y}(-t)$$

We can thus characterize the link between the cdf from the difference of two dependent variables and the cdf from two independent variables. The previous equation can be rewritten as a function of $t$

$$F_{|x-y|}(t) = \int_{-1}^{0} (1 + t)dt + \int_{0}^{t} (1 - t)dt - \int_{-1}^{-t} (1 + t)dt = 2t - t^2 + 1$$

Therefore,

$$f_p(t) = 2(1 - t) \quad (10)$$

Remember that the density function of $q$ is different from 0, $f_q(\frac{w + v}{\alpha}) \neq 0$. It follows that $\frac{w + v}{\alpha} \in [0, 1] \iff w \in [-v, (\alpha - v)]$ and $w \in [0, 1]$. Thus, combining equations (3) and (10), we find

$$f_{n_3}(v) = 2 \int_{0}^{1} f_q(\frac{v + w}{\alpha})(1 - w)\frac{1}{\alpha}dw \quad (11)$$

where $\theta(v, w) = (1 - \frac{w + v}{\alpha})(1 - w)$. Consider then the case $\alpha \geq 1$. First, we can extract the following five different cases:

- $v < -1, f_{n_3} = 0$
- $-1 < v < 0$, thus the pdf is $4 \int_{0}^{1} \theta(v, w)dw = 4\frac{3v^2 + 6\alpha v + 3v^2 - \alpha^2 - 3\alpha - 1}{6\alpha}$;
- $0 < v < \alpha - 1$, thus the pdf is $4 \int_{0}^{1} \theta(v, w)dw = 4\frac{3(\alpha - v) - 1}{6\alpha}$;
- $\alpha - 1 < v < \alpha$, thus the pdf is $4 \int_{0}^{\alpha - v} \theta(v, w)dw = 4\frac{-v^3 + 3\alpha^2 v + 3v^2 - 3\alpha^2 - 6\alpha + 3v^2 + 3\alpha}{6\alpha^2}$;
- $v > \alpha, f_{n_3} = 0$

Second, we characterize $F_{n_3}$ in the different cases derived above.

- $v < -1, F_{n_3}(v) = 0$;
- $-1 < v < 0, F_{n_3}(v) = 4\frac{4v^3 + 12\alpha v^2 + 12\alpha v + 4\alpha - v^3 - 4v^2 - 6v^2 - 4v - 1}{6\alpha}$. 

12
\[ F_{\pi B}(v) = \begin{cases} \frac{12\alpha v^4 + 4\alpha^2 - 6\alpha^2 - 4v - 1}{6\alpha^2}, & 0 < v < \alpha - 1, \\ \frac{\alpha v^4 - 4\alpha^2 v - 2\alpha^2 - 4\alpha^2 + 12\alpha^2 v + 6\alpha^2 - 12\alpha^2 + 4\alpha^2}{6\alpha^2}, & \alpha - 1 < v < \alpha, \\ \frac{\alpha v^4}{6\alpha^2}, & v > \alpha. \end{cases} \]

Finally, \( P_{\text{eff},1} \) is such that

\[
P_{\text{eff},1} = \frac{1}{2} \times 1 + \frac{1}{2} (1 - P(\alpha \Delta q - \Delta p \leq 0))
= \frac{1}{2} + \frac{1}{2} (1 - F_{\pi B}(0))
= \frac{1}{2} (2 - \frac{4\alpha - 1}{6\alpha^2}).
\]

### 4.3 Efficiency when 2 bidders remain \( (P_{\text{eff},2}) \)

We follow here the same type of reasoning taking into account that \( q = \Delta q \) and \( p = \Delta p \) are both \( \leq 0 \). Let \( p = -|x - y| \) where \( x \) and \( y \) are two uniformly distributed random variables on \([0, 1]\). The cdf is then with \( t < 0 \)

\[
F_{-x-y}(t) = P(-|x - y| \leq t) = P(|x - y| \geq t) = 1 - P(|x - y| \leq t)
= 1 - P(t \leq x - y \leq t)
= 1 - (F_{x-y}(-t) - F_{x-y}(t))
= 1 - \left( \int_{-t}^{t} (1 - t)dt - \frac{1}{2} - \int_{0}^{t} (1 + t)dt \right)
= t^2 + 2t + 3
\]

Therefore,

\[
f_{p}(t) = 2(1 + t)
\]

Characterizing the pdf and the cdf when \( \alpha \geq 1 \), we end up with

\[
P_{\text{eff},2} = \frac{1}{2} (2 - \frac{6\alpha^2 - 4\alpha + 1}{6\alpha^2})
\]

### 4.4 RAQ efficiency and the decision rule

The RAQ efficiency is thus

\[
P_{\text{eff}}(K, \alpha) = P(q_1 > K \text{ and } q_2 > K)P_{\text{eff},1} + P(q_1 > K \text{ and } q_2 \leq K)P_{\text{eff},2}
= K(1 - K) \left( 2 - \frac{4\alpha - 1}{6\alpha^2} \right) + \frac{(1 - K)^2}{2} \left( 2 - \frac{6\alpha^2 - 4\alpha + 1}{6\alpha^2} \right)
\]
In Figure 4, for a same realization of \( K \), efficiency is quite stable across the different values of \( \alpha \). The optimal choice of \( K \) values has already been adressed in section 3.3. This decision rule can be enriched if we also take account the probability of efficiency. In figure 5, a parametric plot has been computed for values of \( K \) ranging from 0 to 1 and for a value of \( \alpha \) equal to 2. If we follow the decision rule proposed in section 3.3, we get a value of \( K \) which is equal to 0.53 (point A). The figure shows that conceding values on the optimal expected revenue difference can increase the probability of efficiency drastically (point B).
5 Conclusions and directions for future research

The aim of this paper is to compare two possible implementations to conduct a multi-attribute procurement auction. The first one, referred to as a reversed multi-attribute auction, forces the buyer to reveal her utility function. The second one, referred to as a Reversed Auction with a Quality threshold, works in two steps. Bidders satisfying a minimum quality threshold are first qualified. Afterwards, competition is restricted to price. The main advantage of the latter mechanism lies in the fact that the buyer has to reveal less information than in the first one.

First, we quantify the expected utility loss of choosing a RAQ instead of a MAA based on the utility maximization objective by establishing the expected revenue difference when only two bidders participate in the procurement. Obviously, the reversed multi-attribute auction leads to better expected utilities but requires more information. Therefore, results have to be interpreted as being the expected premium to pay for choosing the second implementation instead of the first one. This analytical result has allowed us to determine values of $K$ that minimize the premium given the importance the buyer assigns to the quality. Moreover, we have shown that the distinction between RAQ and traditional reversed auction based on price is only meaningful if $\alpha$ exceeds a given threshold. In addition, numerical simulations have led us to extend these observations to the case when more than two bidders participate in the auction. Finally, the question of efficiency has been addressed.

Of course, a number of questions remain to be explored. First, the impacts of assumptions such as the independence between price and quality attributes or the linear form of utility functions have to be weakened. Moreover, the density probability function of the revenue difference has to be established in order to further refine the results based on its expected value. Finally, the question of choosing the ideal value of the parameter $K$ has only been sketched. In practice, the decision process involves not only the minimization of the premium but also other aspects such as, for instance, the minimization of the zero participation probability or the maximization of the minimum quality threshold itself. This multicriteria problem deserves more attention.

References


6 Appendix A

To be complete, we characterize the MAA implementation under the hypothesis that \( \alpha \leq 1 \). Again, we extract the following five cases:

- \( v < -1, f_{\pi B}(v) = 0 \);
- \( -1 \leq v < \alpha - 1 \), thus the pdf is \( \int_{-1}^{v} \frac{1}{a} dw = \frac{1}{a} \); 
- \( \alpha - 1 \leq v < 0 \), thus the pdf is \( \int_{\alpha - 1}^{v} \frac{1}{a} dw = 1 \);
- \( 0 \leq v < \alpha \), thus the pdf is \( \int_{0}^{\alpha - v} \frac{1}{a} dw = \frac{a - v}{a} \);
- \( v \geq \alpha, f_{\pi B}(v) = 0 \)

Second, we characterize \( F_{\pi B} \) in the different cases given above.

- \( v < -1, F_{\pi B}(v) = 0 \);
- \( -1 \leq v < \alpha - 1 \), \( F_{\pi B}(v) = 0 + \int_{-1}^{0} \frac{v}{a} dx = \frac{1}{2a}(v + 1)^2 \);
- \( \alpha - 1 \leq v < 0, F_{\pi B}(v) = \frac{1}{2a}(v + 1)^2 \left|_{v=\alpha-1}^{v} \right. + \int_{\alpha - 1}^{v} \frac{1}{a} dx = 1 - \frac{v}{2} + v \);
- \( 0 \leq v < \alpha, F_{\pi B}(v) = (1 - \frac{v}{2} + v) \left|_{v=0}^{v} \right. + \int_{0}^{\alpha - v} \frac{a-x}{a} dx = 1 + v - \frac{v^2}{2a} - \frac{v}{2} \);
- \( v \geq \alpha, F_{\pi B}(v) = \left| F_{\pi B}(v) \right|_{v=\alpha} \) = 1;

Third, we are now ready to derive the joint cumulative distribution function and finally the pdf. Let \( X \) and \( Y \) be two variables of distribution \( F_X \) and \( F_Y \) respectively. Denote \( Z = \min(X,Y) \). \( Z \) is thus the minimum utility the firm \( B \) received as the outcome of the second implementation. The cdf of \( Z \) is thus \( F_Z(t) = P(Z \leq t) = P(\min(X,Y) \leq t) = 1 - P(\min(X,Y) > t) = 1 - P(X > t \text{ and } Y > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - F_X(t))(1 - F_Y(t)) \). From Assumption 3, it turns that

\[
F_Z(t) = 1 - (1 - 2F(t) + F^2(t)) = F(t)(2 - F(t)) \hspace{1cm} (15)
\]

Therefore, the pdf \( f_Z(t) \) is

\[
f_Z(t) = f(t)(2 - F(t)) + F(t)(-f(t)) = 2f(t)(1 - F(t)) \hspace{1cm} (16)
\]

Let summarize \( F_{\pi B}, f_{\pi B} \) and \( f_Z \) in the following table the different cases:
Finally, we can compute the expected payoff of firm B. The three relevant cases are

\[
\begin{align*}
(a) & \quad \frac{1}{\alpha^2} \int_{-1}^{\alpha-1} (v + 1)(2\alpha - (v + 1)^2)vdv = \frac{1}{60}(-12\alpha^3 + 55\alpha^2 - 60\alpha) \\
(b) & \quad \frac{1}{\alpha^2} \int_{0}^{\alpha-1} (\alpha - 2v)vdv = \frac{\alpha^3}{6} - \alpha^2 + \frac{3}{2}\alpha - \frac{2}{3} \\
(c) & \quad \frac{1}{\alpha^2} \int_{0}^{\alpha} (\alpha - v)^2vdv = \frac{1}{20}\alpha^3
\end{align*}
\]

Thus, (a)+(b)+(c) is

\[E[\pi_B] = \frac{1}{60} (\alpha^3 - 5\alpha^2 + 30\alpha - 40) \quad (17)\]
7 Appendix B

Expected difference for several quality values of $\alpha$

Figure 6: This Figure contains four panels $E[\text{diff}]$ with respect to different values of $\alpha$ labelled (a), (b), (c), and (d).
We present here the general form of the RAQ implementation with \( n \) bidders. In this framework, \( n + 1 \) different cases may occur. It represents a range from all bidders are qualified to no bidder is qualified. Nonetheless, we can group these \( n + 1 \) cases into three main categories.

- **Category 1**: more than one bidder is qualified for the auction. \( \forall i \in [0, n - 2] \), the probability \( P \) of such an event is

\[
P = \binom{n}{n-i}(1-K)^{n-i}K^i
\]

where \( i = 0 \) represents the case in which \( n \) bidders are qualified for the final round and \( i = n - 2 \) in which only two bidders remain. The payoff is \( \pi_{B|q=K} = aK - p(2) \) where \( p(2) \) represents the second lowest price among \( n - i \) offers.

- **Category 2**: only one bidder is qualified for the auction. The probability of such an event is:

\[
P(q_1 > K \text{ and } q_2 \leq K) = 1 - (1-K)^2 - K^2 = 2K(1-K)
\]

The payoff is \( \pi_B = aK - 1 \)

- **Category 3**: no one is qualified. The probability of such an event is:

\[
P(q_1 \leq K \text{ and } q_2 \leq K) = P(q_1 \leq K)P(q_2 \leq K) = F_q(K)^2 = K^2
\]

The payoff is \( -1 \) as we consider that firm \( B \) will face a loss not having the service fulfilled (in the example, the transport will not take place and thus, the firm \( B \) will face extra storage costs for example)

Consider a random variable \( Z \) with the second lowest price among \( n - i \) offers \( \forall i \in [0, n - 2] \). The c.d.f. yields the following

\[
F_Z(t) = \sum_{k=0}^{n-i-2} \binom{n-i}{k}(t)^{n-i-k}(1-t)^k
\]

The expectation becomes \( \forall i = 0, ..., n - 2 \)

\[
E[Z] = \int_0^1 f_Z(t)dt = \int_0^1 (1-t)^{n-i-k}(t)^{n-i-k}dt
\]

The expected payoff from buyer \( B \) is

\[
E[\pi_{B|q=K}] = \sum_{j=0}^{n} (aQ_j - E_j)P_j
\]

where

\[
Q_j = \begin{cases} 
K & \forall j = 0, 1, ..., n - 1 \\
0 & \forall j = n
\end{cases}
\]

\[
E_k = \begin{cases} 
E[Z] & \forall k = 0, 1, ..., n - 2 \\
1 & \forall k = n - 1, n
\end{cases}
\]

\[
P_I = \binom{n}{n-I}(1-K)^{n-k}K^k \forall I = 0, ..., n
\]